# The max cut problem

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# Max Cut

| Introduction  | - Given a graph $G(V E)$ , find $S \subset V$ such that the cut                                  |
|---|--|
| <ul> <li>Max Cut</li> </ul>   | Siven a graph $O(V, E)$ , a mu $D \subseteq V$ such that the cut                                 |
| An application in<br>statistical physics<br>The Goemans and<br>Williamson algorithm | $ V  = n$ with weights $w_{ij}$ for $ egree S, V \setminus S$ is maximum.<br>- all $(i,j) \in E$ |

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Introducing the variables:

$$x_i = \left\{ \begin{array}{ll} 1 & \text{if } i \in S \\ -1 & \text{if } i \in V \setminus S \end{array} \right.$$

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| Introduction                            | - Civon a            |
|---|----------------------|
| Max Cut                                 | Given a              |
| An application in statistical physics   | $ V  = n \mathbf{v}$ |
| The Goemans and<br>Williamson algorithm | all $(i, j) \in$     |

en a graph G(V, E), rightarrow find  $S \subseteq V$  such that the cut  $i, j) \in E$ 

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Introducing the variables:

$$x_i = \left\{ \begin{array}{ll} 1 & \text{ if } i \in S \\ -1 & \text{ if } i \in V \setminus S \end{array} \right.$$

The max cut problem can be expressed as

$$\max \frac{1}{2} \sum_{i < j} w_{ij} (1 - x_i x_j)$$
  
$$x_i^2 = 1 \quad i = 1, \dots, n$$
 (MC)

# Max Cut

all  $(i, j) \in E$ 

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|-----------------------------|
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Introducing the variables:

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \in V \setminus S \end{cases}$$

Given a graph G(V, E), rightarrow find  $S \subseteq V$  such that the cut |V| = n with weights  $w_{ij}$  for rightarrow  $S, V \setminus S$  is maximum.

The max cut problem can be expressed as

$$\max \frac{1}{2} \sum_{i < j} w_{ij} (1 - x_i x_j)$$
  
$$x_i^2 = 1 \quad i = 1, \dots, n$$
 (MC)

It can be rewritten as

$$\max_{i} x^{T} C x$$

$$x_{i}^{2} = 1 \quad i = 1, \dots, n$$
(MC)

where  $C = \frac{1}{4} \left( \text{Diag}(We) - W \right) = \frac{1}{4}L$  (*L* is the Laplacian of the graph)

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A very interesting problem in statistical physics (within theory of magnetism) is the determination of ground states of spin glasses

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A spin glass is an alloy of magnetic impurities diluited in a non magnetic metal

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A spin glass is an alloy of magnetic impurities diluited in a non magnetic metal

A peak in magnetic susceptibility at a certain temperature means a phase transition. It is not known what happens at low temperature.

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A very interesting problem in statistical physics (within theory of magnetism) is the determination of ground states of spin glasses

- A spin glass is an alloy of magnetic impurities diluited in a non magnetic metal
- A peak in magnetic susceptibility at a certain temperature means a phase transition. It is not known what happens at low temperature.
- There are some theories on spin glasses behavior but it is impossible to realize by practical experiments the situation predicted by the theory, so mathematical models are used

| Introduction   | - Assume a spin glass contains $n$ magnetic impurities (atoms). |
|--|---|
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Assume a spin glass contains n magnetic impurities (atoms). The magnetic orientation of each atom is represented by  $S_i \in \mathbb{R}^3$ ,  $||S_i|| = 1, i = 1, ..., n$ 

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Assume a spin glass contains n magnetic impurities (atoms). The magnetic orientation of each atom is represented by  $S_i \in \mathbb{R}^3$ ,  $||S_i|| = 1, i = 1, ..., n$ For each pair i, j of atoms there is an interaction  $J_{ij}$  depending on the material and on the distance (decreases quickly with the distance), an example is

$$J_{ij} = A \frac{\cos(Dr_{ij})}{B^3 r_{ij}^3}, \quad A, B, D$$
 depending on the material

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$$J_{ij} = A \frac{\cos(Dr_{ij})}{B^3 r_{ij}^3}, \quad A, B, D$$
 depending on the material

If atom i and j have spins  $S_i$  and  $S_j$ , the energy interaction between the two atoms is given by

$$H_{ij} = J_{ij} \langle S_i, S_j \rangle$$

Given a configuration  $\omega$ , the energy of the system is given by the hamiltonian

$$H(\omega) = -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} \langle S_i, S_j \rangle - h \sum_{i=1}^{n} J_{ij} \langle S_i, F \rangle$$

where  $F \in \mathbb{R}^3$ , ||F|| = 1 represents the orientation of an exterior magnetic field of strength h.

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|---|----|----|---|----|----|----|---|---|
| I | 11 | u  | υ | uι | JC | u  | J |   |

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A possible simplification is to replace  $S_i \in \mathbb{R}^3$ , i = 1, ..., n and  $F \in \mathbb{R}^3$  with  $s_i \in \{-1, 1\}$ , i = 1, ..., n and  $f \in \{-1, 1\}$ , meaning magnetic north pole up and magnetic north pole down

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The short range Ising spin glass model allowed to predict a phase transition that was experimentally observed (Nobel prize to Onsager)

| Introduction   | - A possible simplification                                 |  |
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| An application in statistical physics                        | A possible simplification $s_i \in \{-1, 1\}, i = 1, \dots$ |  |
| <ul> <li>Spin glasses</li> </ul>                             |   |  |
| <ul> <li>Mathematical model</li> </ul>                       | and magnetic north pole                                     |  |
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- 1. Long range model: considering the interaction between all pairs of atoms
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The short range Ising spin glass model allowed to predict a phase transition that was experimentally observed (Nobel prize to Onsager)

At 0 K the spin glasses configuration attains a minimum energy configuration, that can be found by minimizing the Hamiltonian associated with the system. This problem can be formulated as a max cut problem

| Introduction   | — We define a graph G with $n+1$ nodes (n atoms plus an external magnet |
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| An application in statistical physics  | represented by the node $0$ )   |
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We define a graph G with n + 1 nodes (n atoms plus an external magnetic field represented by the node 0)

For a pair i, j G contains an edge ij if the interaction between i and j is not zero, and the weight of the edge is equal to  $J_{ij}$ .

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For a pair i, j G contains an edge ij if the interaction between i and j is not zero, and the weight of the edge is equal to  $J_{ij}$ .

There exist n edges 0i with weigth  $J_{0i} = h$ 

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There exist n edges 0i with weigth  $J_{0i} = h$ 

Then the Hamiltonian can be represented as a quadratic function

$$H(\omega) = -\sum_{ij \in E, i,j>0} J_{ij} s_i s_j - h \sum_{i=1}^n s_i = -\sum_{ij \in E} J_{ij} s_i s_j$$

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$$H(\omega) = -\sum_{ij \in E, i,j>0} J_{ij} s_i s_j - h \sum_{i=1}^n s_i = -\sum_{ij \in E} J_{ij} s_i s_j$$

Each spin configuration corresponds to a partition of the nodes into  $V^+ = \{i: s_i = 1\}, V^- = \{i: s_i = -1\}$ 

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For any 
$$W \subseteq V$$
, define  $E(W) = \{ij \in E : i \in W, j \in W\}$ ,  
 $\delta(W) = \{ij \in E : i \in W, j \in V \setminus W\}$ 

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For any 
$$W \subseteq V$$
, define  $E(W) = \{ij \in E : i \in W, j \in W\}$ ,  $\delta(W) = \{ij \in E : i \in W, j \in V \setminus W\}$ 

$$H(\omega) = -\sum_{ij\in E} J_{ij}s_is_j = -\sum_{ij\in E(V^+)} J_{ij} - \sum_{ij\in E(V^-)} J_{ij} + \sum_{ij\in \delta(V^+)} J_{ij}$$

Setting  $C = \sum_{ij \in E} J_{ij}$ , we get

 $\delta(W) = \{ ij \in E : i \in W, j \in V \setminus W \}$ 

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For any  $W \subseteq V$ , define  $E(W) = \{ij \in E : i \in W, j \in W\}$ ,

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 $H(\omega) + C = 2$   $\sum J_{ij}$  $ij \in \delta(V^+)$ 

 $H(\omega) = -\sum J_{ij}s_is_j = -\sum J_{ij} - \sum J_{ij} + \sum J_{ij}$  $ij \in E(V^+)$   $ij \in E(V^-)$   $ij \in \delta(V^+)$ 

 $\delta(W) = \{ ij \in E : i \in W, j \in V \setminus W \}$ 

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Setting  $C = \sum_{ij \in E} J_{ij}$ , we get

 $ij \in E$ 

$$H(\omega) + C = 2 \sum_{ij \in \delta(V^+)} J_{ij}$$

 $H(\omega) = -\sum J_{ij}s_is_j = -\sum J_{ij} - \sum J_{ij} + \sum J_{ij}$ 

 $ij \in E(V^+)$   $ij \in E(V^-)$   $ij \in \delta(V^+)$ 

For any  $W \subseteq V$ , define  $E(W) = \{ij \in E : i \in W, j \in W\}$ ,

Setting  $c_{ij} = -J_{ij}$  for all  $ij \in E$ , then minimizing  $H(\omega)$  is equivalent to solve

$$\max_{V^+ \subseteq V} \sum_{ij \in \delta(V^+)} c_{ij}$$

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 $x^T C x = \langle C, x x^T \rangle = \langle C, X \rangle, \text{ for } X = x x^T$ 

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The max cut problem is then equivalent to:

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 $\max \quad \begin{array}{l} \langle C, X \rangle \\ \operatorname{diag}(X) = e \\ X \succeq 0 \\ \operatorname{rank}(X) = 1 \end{array}$ 

 $x^T C x = \langle C, x x^T \rangle = \langle C, X \rangle, \text{ for } X = x x^T$ 

# Relaxation

| Introduction<br>An application in<br>statistical physics   | Noting that $x^T C x = \langle C, x x^T \rangle = \langle C, X \rangle$ , for $X = x x^T$  |
|--|--|
| The Goemans and<br>Williamson algorithm<br>• Relaxation<br>• Properties of the SDP   | The max cut problem is then equivalent to:   |
| <ul> <li>Alternative way of deriving the SDP relaxation</li> <li>Goemans and</li> <li>Williamson algorithm</li> <li>Performance of the algorithm</li> </ul>              | $ \begin{array}{ll} \max & \langle C, X \rangle \\ & \operatorname{diag}(X) = e \\ & X \succeq 0 \\ & \operatorname{rank}(X) = 1 \end{array} \end{array} $ |
| <ul> <li>Nonlinear<br/>reformulation of max cut<br/>I</li> <li>Nonlinear<br/>reformulation of max cut<br/>II</li> <li>Nesterov's result</li> <li>Bibliography</li> </ul> | Dropping the rank constraint, the standard SDP relaxation of max cut can be obtained:<br>$\max  \langle C, X \rangle$ $\operatorname{diag}(X) = e$         |

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# **Properties of the SDP relaxation**

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- 1. Slater is satisfied for both primal and dual so strong duality holds
- 2. The constraints  $\operatorname{diag}(X) = e$ ,  $X \succeq 0$  imply that  $-1 \leq X_{ij} \leq 1$  for all i, j.
- 3. If X is feasible, and  $|X_{ij}| = 1$ , then  $x_{ik} = \operatorname{sgn}(x_{ij})x_{jk}$
- 4. The only feasible matrix of rank 1 satisfying  $X_{ij} \in \{-1, 1\}$  are of the form  $xx^T$ , with  $x \in \{-1, 1\}^n$

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$$\frac{1}{2} \max \sum_{i < j} w_{ij} (1 - x_i x_j)$$

$$x_i^2 = 1 \quad i = 1, \dots, n$$
(MC)

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$$x_i^2 = 1 \quad i = 1, \dots, n$$
(MC)

$$\begin{array}{ll} x_i & \Rightarrow & v_i \in \mathbb{R}^k, \, k \le n \\ x_i^2 = 1 & \Rightarrow & \|v_i\| = 1 \end{array}$$

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$$\begin{array}{ll} x_i & \Rightarrow & v_i \in \mathbb{R}^k, \, k \le n \\ x_i^2 = 1 & \Rightarrow & \|v_i\| = 1 \end{array}$$

 $v_i$  is the relaxation of  $x_i \in \{-1, 1\}$  to the *n*-dimensional unit sphere.

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The max cut problem can be written as

 $\frac{1}{2} \max \sum_{i < j} w_{ij} (1 - x_i x_j)$   $x_i^2 = 1 \quad i = 1, \dots, n$ (MC)

The feasible region can be enlarged:

 $\begin{array}{ll} x_i & \Rightarrow & v_i \in \mathbb{R}^k, \, k \le n \\ x_i^2 = 1 & \Rightarrow & \|v_i\| = 1 \end{array}$ 

 $v_i$  is the relaxation of  $x_i \in \{-1, 1\}$  to the *n*-dimensional unit sphere. We get the problem

$$\max \sum_{i,j} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$$
$$\|v_i\|^2 = 1 \quad i = 1, \dots, n, v_i \in \mathbb{R}^n$$

(the same as  $X = VV^T$ )

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$$\max \sum_{i,j} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$$
$$\|v_i\|^2 = 1 \quad i = 1, \dots, n, v_i \in \mathbb{R}^n$$

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# $\max \sum_{i,j} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$ $\|v_i\|^2 = 1 \quad i = 1, \dots, n, v_i \in \mathbb{R}^n$

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$$\max \sum_{i,j} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$$
$$\|v_i\|^2 = 1 \quad i = 1, \dots, n, v_i \in \mathbb{R}^n$$

# S2 Choose a random vector h on the unit sphere

S3 Set 
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 $\max \sum_{i,j} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$  $\|v_i\|^2 = 1 \quad i = 1, \dots, n, v_i \in \mathbb{R}^n$ 

S2 Choose a random vector h on the unit sphere

S3 Set  $V = \{i : h^T v_i \ge 0\}$ 

Let  $\boldsymbol{W}$  be the corresponding cut

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1. The expected value of the produced cut is:

$$E[W] = \sum_{i < j} w_{ij} \frac{\arccos(v_i^T v_j)}{\pi}$$

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1. The expected value of the produced cut is:

$$E[W] = \sum_{i < j} w_{ij} \frac{\arccos(v_i^T v_j)}{\pi}$$

2. Assume  $w_{ij} \ge 0$ . Then  $E(W) \ge \alpha \frac{1}{2} \sum_{i < j} w_{ij} (1 - v_i^T v_j)$ , with

$$\alpha = \min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos(\theta)}$$

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**3.**  $\alpha > 0.87856$ 

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$$\alpha = \min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos(\theta)}$$

**3**. 
$$\alpha > 0.87856$$

4. Assume 
$$w_{ij} \ge 0$$
. Then  $\frac{z_{\rm MC}^*}{z_{\rm MC}^* - z_{\rm SDP}^*} > 0.87856$ 

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$$\alpha = \min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos(\theta)}$$

**3**. 
$$\alpha > 0.87856$$

- 4. Assume  $w_{ij} \ge 0$ . Then  $\frac{z_{\rm MC}^*}{z_{\rm MC}^* z_{\rm SDP}^*} > 0.87856$
- 5. Note that there is no polynomial approximation algorithm with constant < 0.9412 unless P=NP [Hästad 1997].

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 $\max \langle C\sigma(Vu), \sigma(Vu) \rangle$  $\|v_i\| = 1 \quad i = 1, \dots, n$  $\|u\| = 1$ 

where for any 
$$a \in \mathbb{R}^n$$
,  $\sigma(a) = \operatorname{sgn}(a)$ , and  $V = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}$ 

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It is also equivalent to

 $\max E_u(\langle C\sigma(Vu), \sigma(Vu) \rangle)$  $\|v_i\| = 1 \quad i = 1, \dots, n$ 

$$||v_i|| = 1$$
  $i = 1, \dots, n$   
 $||u|| = 1$ 

 $\max \langle C\sigma(Vu), \sigma(Vu) \rangle$ 

where for any 
$$a \in \mathbb{R}^n$$
,  $\sigma(a) = \operatorname{sgn}(a)$ , and  $V = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}$ 

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$$\max \frac{2}{\pi} \langle C, \arcsin(X) \rangle$$
  
$$\operatorname{diag}(X) = e$$
  
$$X \succeq 0$$

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The max cut problem can be rewritten as

$$\max \frac{2}{\pi} \langle C, \arcsin(X) \rangle$$
  
$$\operatorname{diag}(X) = e$$
  
$$X \succeq 0$$

Sketch of the proof: Choose  $V = X^{\frac{1}{2}}$ . Then we need to prove

$$E_u(\langle C\sigma(Vu), \sigma(Vu) \rangle) = \frac{2}{\pi} \langle C, \arcsin(X) \rangle$$

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$$\frac{z_{\rm MC}^*}{z_{\rm MC}^* - z_{\rm SDP}^*} > \frac{2}{\pi} = 0.6366$$

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 $\frac{z_{\rm MC}^*}{z_{\rm MC}^* - z_{\rm SDP}^*} > \frac{2}{\pi} = 0.6366$ 

Define  $X^{\circ k} = X \circ X^{\circ (k-1)}$  and consider that

$$\operatorname{arcsin}(X) = X + \frac{1}{2} \frac{X^{\circ 3}}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{X^{\circ 5}}{5} + \dots$$

$$\frac{2}{\pi}\langle C, \arcsin(X) - X \rangle \ge 0$$

$$z_{MC}^* \ge \frac{2}{\pi} \langle C, \arcsin(X) \rangle \ge \frac{2}{\pi} \langle C, X \rangle = \frac{2}{\pi} z_{SDP}^* \ge \frac{2}{\pi} z_{MC}^*.$$

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