The max cut problem

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Given a graph G(V,E), $|V| = n$ with weights w_{ij} for all $(i,j) \in E$ ⇒ find $S \subseteq V$ such that the cut $S, V \setminus S$ is maximum.

Introducing the variables:

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \in V \setminus S \end{cases}$$

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It can be rewritten as

$$\begin{aligned} &\max x^T C x \\ &x_i^2 = 1 \quad i = 1, \dots, n \end{aligned} \tag{MC}$$

where $C = \frac{1}{4}\left(\mathrm{Diag}(We) - W\right) = \frac{1}{4}L$ (L is the Laplacian of the graph)

Spin glasses

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There are some theories on spin glasses behavior but it is impossible to realize by practical experiments the situation predicted by the theory, so mathematical models are used

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If atom i and j have spins S_i and S_j, the energy interaction between the two atoms is given by

$$H_{ij} = J_{ij}\langle S_i, S_j\rangle$$

Given a configuration ω, the energy of the system is given by the hamiltonian

$$H(\omega) = -\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} J_{ij}\langle S_i, S_j\rangle - h\sum_{i=1}^{n} J_{ij}\langle S_i, F\rangle$$

where $F \in \mathbb{R}^3$, $\|F\| = 1$ represents the orientation of an exterior magnetic field of strength h.

Ising spins

A possible simplification is to replace $S_i \in \mathbb{R}^3$, $i = 1, \ldots, n$ and $F \in \mathbb{R}^3$ with $s_i \in \{-1, 1\}$, $i = 1, \ldots, n$ and $f \in \{-1, 1\}$, meaning magnetic north pole up and magnetic north pole down

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At 0 K the spin glasses configuration attains a minimum energy configuration, that can be found by minimizing the Hamiltonian associated with the system. This problem can be formulated as a max cut problem

Ising model and max cut I

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Then the Hamiltonian can be represented as a quadratic function

$$H(\omega) = -\sum_{ij \in E,\, i,j>0} J_{ij} s_i s_j - h \sum_{i=1}^{n} s_i = -\sum_{ij \in E} J_{ij} s_i s_j$$

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Each spin configuration corresponds to a partition of the nodes into $V^+ = \{i : s_i = 1\}, V^- = \{i : s_i = -1\}$

Ising model and max cut II

For any $W \subseteq V$, define $E(W) = \{ij \in E : i \in W, j \in W\}$, $\delta(W) = \{ij \in E : i \in W, j \in V \setminus W\}$

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Setting $c_{ij} = -J_{ij}$ for all $ij \in E$, then minimizing $H(\omega)$ is equivalent to solve

$$\max_{V^+ \subseteq V} \sum_{ij \in \delta(V^+)} c_{ij}$$

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Noting that

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Dropping the rank constraint, the standard SDP relaxation of max cut can be obtained:

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Properties of the SDP relaxation

1. Slater is satisfied for both primal and dual so strong duality holds
2. The constraints $\mathrm{diag}(X) = e$, $X \succeq 0$ imply that $-1 \leq X_{ij} \leq 1$ for all i, j.
3. If X is feasible, and $|X_{ij}| = 1$, then $x_{ik} = \mathrm{sgn}(x_{ij})x_{jk}$
4. The only feasible matrix of rank 1 satisfying $X_{ij} \in \{-1, 1\}$ are of the form xx^T, with $x \in \{-1, 1\}^n$

Alternative way of deriving the SDP relaxation

The max cut problem can be written as

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$$\begin{array}{lcl} x_i & \Rightarrow & v_i \in \mathbb{R}^k,\ k \leq n \\ x_i^2 = 1 & \Rightarrow & \|v_i\| = 1 \end{array}$$

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$$\begin{aligned} &\max\sum_{i,j}\sum_{i<j} w_{ij}(1 - v_i^T v_j) \\ &\|v_i\|^2 = 1 \quad i = 1,\dots,n, v_i \in \mathbb{R}^n \end{aligned}$$

(the same as $X = VV^T$)

Goemans and Williamson algorithm

S1 First, solve

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Let W be the corresponding cut

Performance of the algorithm

1. The expected value of the produced cut is:

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$$\alpha = \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos(\theta)}$$

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4. Assume $w_{ij} \geq 0$. Then $\frac{z^*_{\mathrm{MC}}}{z^*_{\mathrm{MC}} - z^*_{\mathrm{SDP}}} > 0.87856$
5. Note that there is no polynomial approximation algorithm with constant < 0.9412 unless P=NP [Hästad 1997] .

Nonlinear reformulation of max cut I

The max cut problem is equivalent to

$$\begin{aligned} &\max \langle C\sigma(Vu), \sigma(Vu) \rangle \\ &\|v_i\| = 1 \quad i = 1, \ldots, n \\ &\|u\| = 1 \end{aligned}$$

where for any $a \in \mathbb{R}^n$, $\sigma(a) = \operatorname{sgn}(a)$, and $V = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}$

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It is also equivalent to

$$\begin{aligned} &\max E_u(\langle C\sigma(Vu), \sigma(Vu)\rangle) \\ &\|v_i\| = 1 \quad i = 1, \ldots, n \end{aligned}$$

Nonlinear reformulation of max cut II

The max cut problem can be rewritten as

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&\max \frac{2}{\pi}\langle C, \arcsin(X)\rangle \\
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Sketch of the proof: Choose $V = X^{\frac{1}{2}}$. Then we need to prove

$$E_u(\langle C\sigma(Vu), \sigma(Vu)\rangle) = \frac{2}{\pi}\langle C, \arcsin(X)\rangle$$

Nesterov's result

If $C \succeq 0$, then

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Define $X^{\circ k} = X \circ X^{\circ (k-1)}$ and consider that

$$\arcsin(X) = X + \frac{1}{2}\frac{X^{\circ 3}}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{X^{\circ 5}}{5} + \dots$$

and $-1 \leq X_{ij} \leq 1$ then $\arcsin(X) - X \succeq 0$. Since $C \succeq 0$, we get

$$\frac{2}{\pi}\langle C, \arcsin(X) - X\rangle \geq 0$$

and hence

$$z^*_{MC} \geq \frac{2}{\pi}\langle C, \arcsin(X)\rangle \geq \frac{2}{\pi}\langle C, X\rangle = \frac{2}{\pi}z^*_{SDP} \geq \frac{2}{\pi}z^*_{MC}.$$

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