SpeeDP: a fast method for solving the SDP relaxation of Max Cut

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LRSDP relaxation

$$\begin{aligned} \min \quad & \langle Q, X \rangle \\ & \operatorname{diag}(X) = e \\ & X \succeq 0, X \in \mathcal{S}^n \end{aligned} \qquad (\mathrm{SDP_{MC}})$$

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$\widehat{r} = \max\{k \in N : k(k+1)/2 \le n\} = \left\lfloor \frac{\sqrt{1+8n}-1}{2} \right\rfloor$ **computable** upper bound on r

Relaxations

(NLP$_r$)

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A NS G.O. condition

Proposition [Grippo, Palagi, P. 09] A point $V^* \in \mathbb{R}^{n \times r}$ is a global minimizer of Problem (NLP$_r$) that solves Problem (SDP$_{\mathrm{MC}}$) if and only if there exists a $\lambda^* \in \mathbb{R}^n$ such that

$$\begin{aligned} &(Q + \Lambda^*)\, V^* = 0 \\ &Q + \Lambda^* \succeq 0 \\ &\langle E_{ii}, V^* \rangle = 1, \quad i = 1, \ldots, n. \end{aligned} \tag{1}$$

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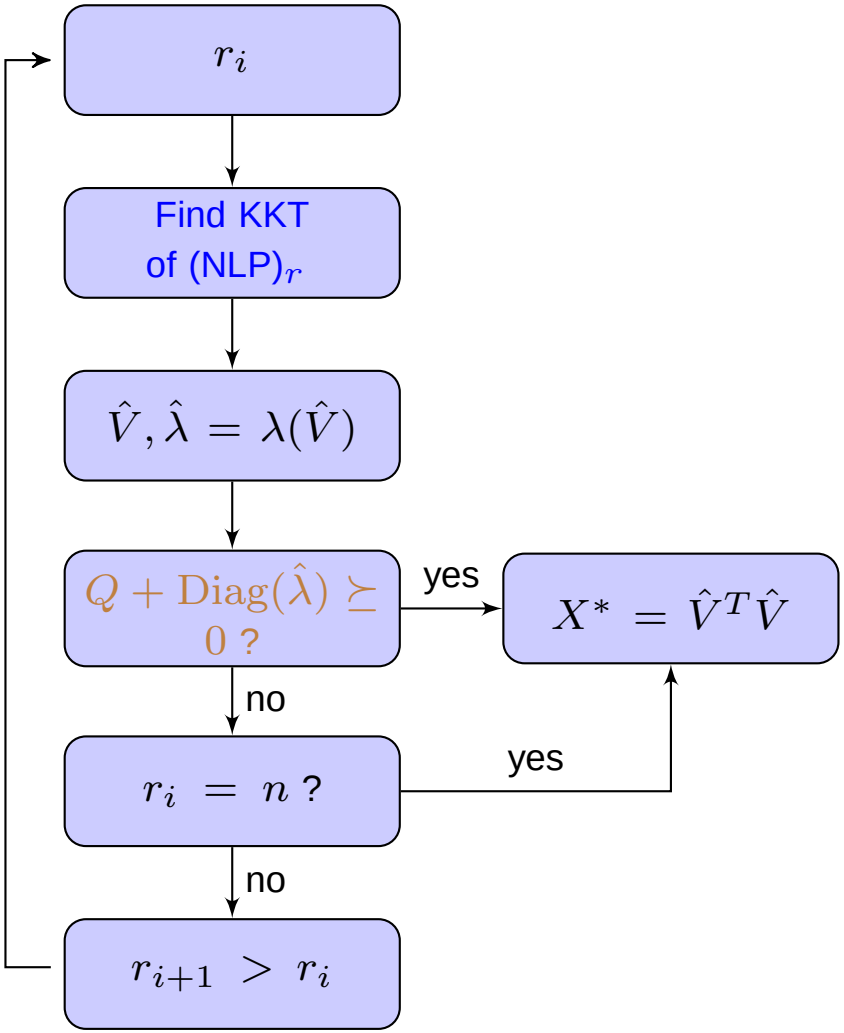
- $\sum_{i=1}^{n} \hat{\lambda}_i = e^T \lambda = -\langle Q, \hat{V}\hat{V}^T \rangle = -q_r(\hat{V})$

The closed expression of the multiplier can be generalized for problems satisfying $A_i A_j = 0$[JourneèBachAbsilSepulchre08]

A general algorithmic approach



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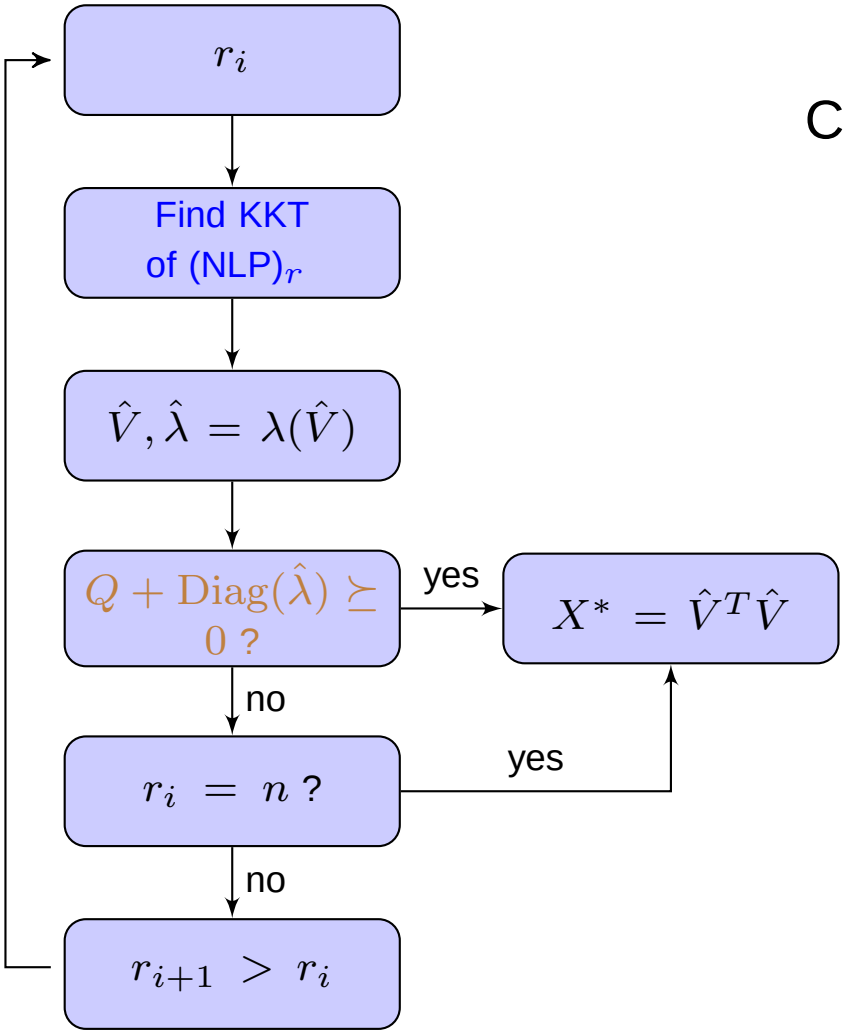

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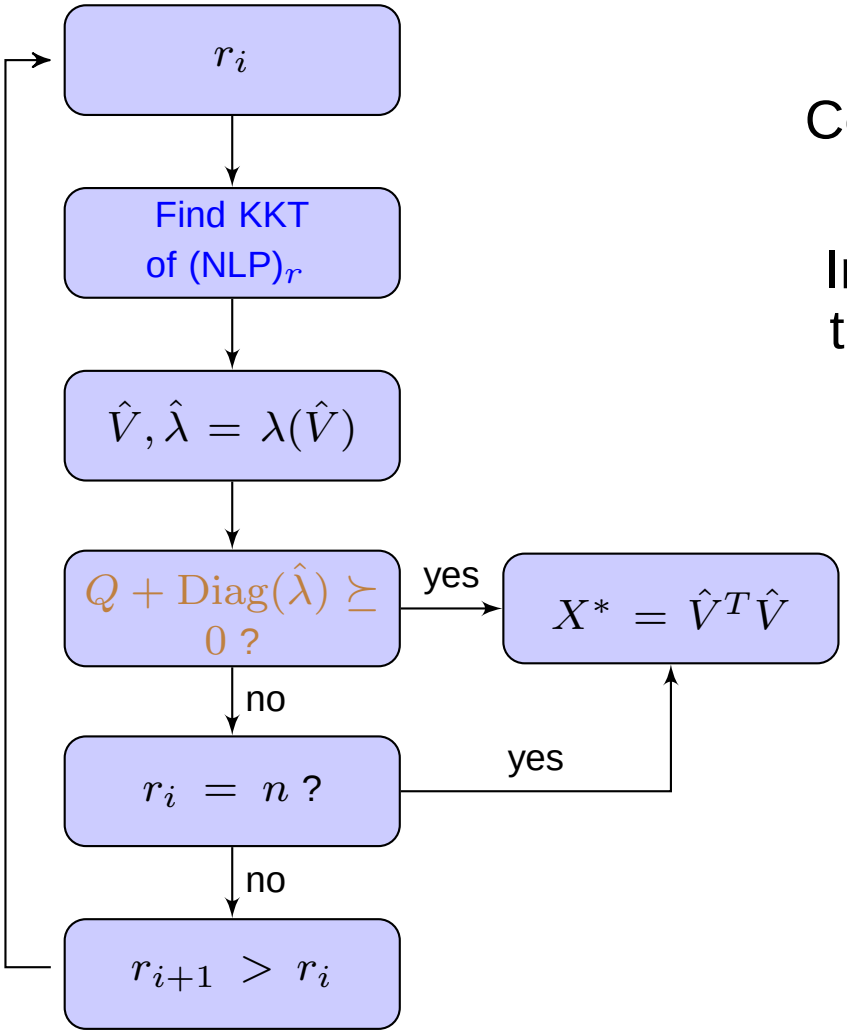

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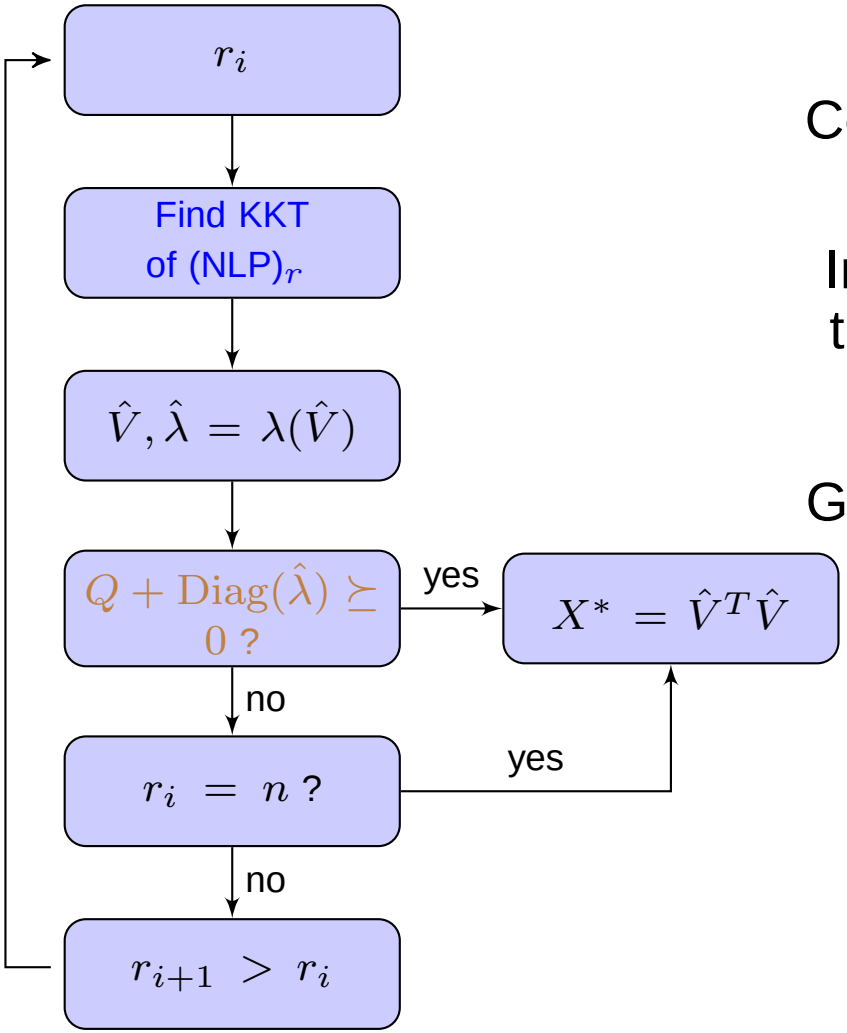

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Global condition: find the minimum eigenvalue of $Q + \mathrm{Diag}(\hat{\lambda})$

Unconstrained formulations for (NLP$_r$)

Augmented Lagrangian [Burer-Monteiro, 2003]

$$\min_V L_a(V, \lambda^k, \epsilon^k) = L(V, \lambda^k) + \frac{1}{\epsilon^k} \sum_i^n \left(\langle E_{ii}, VV^T \rangle - 1 \right)^2$$

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Exact Penalty Function [Grippo-Palagi-P., 2009]

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Quotient-Type Function: using the change of variables $X_{ij} = (v_i/\|v_i\|)^T v_j/\|v_j\|$, $v_i \in \mathbb{R}^r, i = 1, \ldots, n$ (note :$V = (v_1 \ \ldots \ v_n)^T$)

$$\min_v \sum_{i,\ j} q_{ij} \frac{v_i^T v_j}{\|v_i\| \|v_j\|}$$

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2. The quotient function is not defined if any $v_i = 0$, but any gradient type method ensures v_i^k away from 0
3. However, it has unbounded level sets, so that standard convergence theory does not apply

A new unconstrained formulation of problem (NLP$_r$)

We propose a modification of function $q_r(v)$:

$$f_\epsilon(v) := \sum_{i,\ j} q_{ij} \frac{v_i^T v_j}{\|v_i\|\|v_j\|} + \frac{1}{\epsilon} \sum_{i=1}^{n} \frac{(\|v_i\|^2 - 1)^2}{d(v_i)},$$

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For a fixed $\varepsilon > 0$ we consider the unconstrained minimization problem

$$\min_{v \in S_\delta} \ f_\epsilon(v), \tag{RQ$_r$}$$

where the open set S_δ is defined as

$$S_\delta := \{v \in \mathbb{R}^{nr} \ : \ \|v_i\|^2 > 1 - \delta, \quad i = 1, \ldots, n\}.$$

Properties of problem ($\mathrm{RQ_r}$)

$$\begin{array}{l} \min f_\epsilon(v), \\ v \in S_\delta \\ v_i \in \Re^r \; i = 1, \ldots, n \end{array} \qquad (\mathrm{RQ_r}) \qquad \begin{array}{l} \min \langle Q, VV^T \rangle \\ \langle E_{ii}, VV^T \rangle = 1, i = 1, \ldots, n \\ V \in \mathbb{R}^{n \times r} \end{array} \qquad (\mathrm{NLP_r})$$

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Problem (NLP_r) can be solved by solving problem ($\mathrm{RQ_r}$) by any standard minimization algorithm

A specific algorithm for problem ($\mathrm{RQ_r}$) I

$$\begin{array}{ll} \min & \sum_{i,\ j} q_{ij} \frac{v_i^T v_j}{\|v_i\|\|v_j\|} + \frac{1}{\epsilon} \sum_{i=1}^{n} \frac{(\|v_i\|^2-1)^2}{d(v_i)} \\ & v \in S_\delta \\ & v_i \in \Re^r \ i = 1, \ldots, n \end{array} \qquad (\mathrm{RQ_r})$$

The barrier term $d(v_i) := \delta^2 - \left(1 - \|v_i\|^2\right)_+^2$, $0 < \delta < 1$ plays a key role to make standard optimization method be globally convergent for problem (RQ_r).

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Nevertheless, a barrier term affects negatively the performance behavior of any optimization method, especially when the produced sequence gets closer to the boundary of S_δ.

A specific algorithm for problem ($\mathrm{RQ_r}$) II

$$\begin{array}{ll} \min & \sum_{i,\ j} q_{ij} \frac{v_i^T v_j}{\|v_i\|\|v_j\|} + \frac{1}{\epsilon} \sum_{i=1}^{n} \frac{(\|v_i\|^2-1)^2}{d(v_i)} \\ & v \in S_\delta \\ & v_i \in \Re^r \ i = 1, \ldots, n \end{array} \qquad (\mathrm{RQ_r})$$

Consider a gradient type iteration, starting from v_0 such that $\|v_i^0\| = 1$ for $i = 1, \ldots, n$:

$$v_i^{k+1} = v_i^k - \alpha^k \nabla_{v_i} f_\varepsilon(v^k) \quad i = 1, \ldots, n$$

where $\alpha^k > 0$ is obtained by a suitable linesearch procedure satisfying at least

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Then, there exists $\bar{\epsilon} > 0$ such that, for any $\epsilon \geq \bar{\epsilon}$, we have for $k = 1, 2, \ldots$

$$\|v_i^k\| \geq 1, \quad i = 1, \ldots, n.$$

and hence $d(v_i) = \delta^2$

Convergence result

Let $\{v^k\}$ generated with a Gradient Method(Non-Monotone), starting from v_0 such that $\|v_i^0\| = 1$ for $i = 1, \ldots, n$. Then, for $\epsilon \geq \bar{\epsilon}$

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We use a a Fortran 90 implementation of the non monotone Barzilai-Borwein gradient method proposed in [Grippo, Sciandrone, COAP2002]

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Step 3 Compute the minimum eigenvalue $\mu_{\min}(\hat{v})$ of $Q + \mathrm{Diag}(\lambda(\hat{v}))$. If $\mu_{\min}(\hat{v}) = 0$ or $j = p$, then stop and return $\hat{v} \in \mathbb{R}^{nr^j}$ and $\mu_{\min}(\hat{v})$, otherwise set $j = j + 1$, and go to Step 1.

Some Remarks

- If $r = n$ problem (NLP$_r$) has no local minimum points [M. Journee, F. Bach, P.A. Absil, R. Sepulchre 2008]

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- Given any $\lambda \in \Re^n$ the point $u = \lambda + \lambda_{\min}(Q + \Lambda)e$ is feasible for problem (DUAL$_{\mathrm{MC}}$), and hence $e^T u = e^T \lambda + n\lambda_{\min}(Q + \Lambda)$ is a bound on the optimal value of (SDP$_{\mathrm{MC}}$).

Implementation

We select the value r^1 according to an heuristic based on n

n	r^1	$\widehat{r}$
≤ 200	8	≤ 19
$\geq 250, < 800$	10	< 39
$\geq 800, < 1000$	15	< 44
$\geq 1000, \leq 5000$	18	< 99
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   - same algorithmic idea of SpeeDP but using an exact penalty function (state of the art)

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3. Gset: randomly generated by a machine-independent graph generator *rudy* (19 pb. with $800 \leq n \leq 20000$ and number of edges ranging from about 5,000 to 40,000)

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The higher the method the better, and the efficiency is measured by how fast the method reaches the value of 1 (since all the methods solve all the problems, all the methods reach the performance value 1 allowing a sufficiently large τ).

Comparison with other codes: cpu time

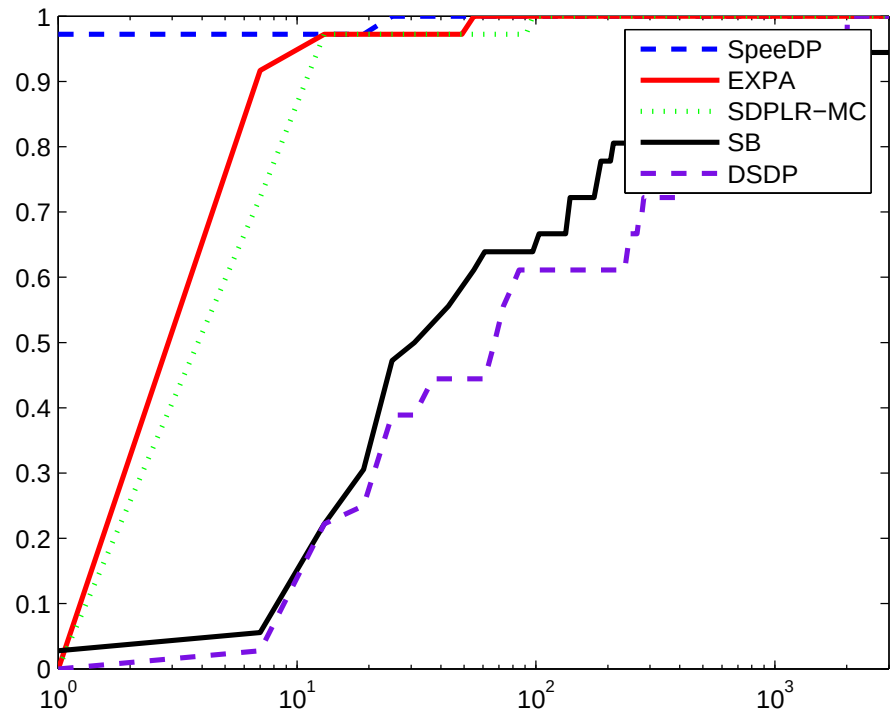

Figure 1: Comparison between NLP based methods, SB and DSDP on the problems up to 10000 variables

Comparison with LR: cpu time

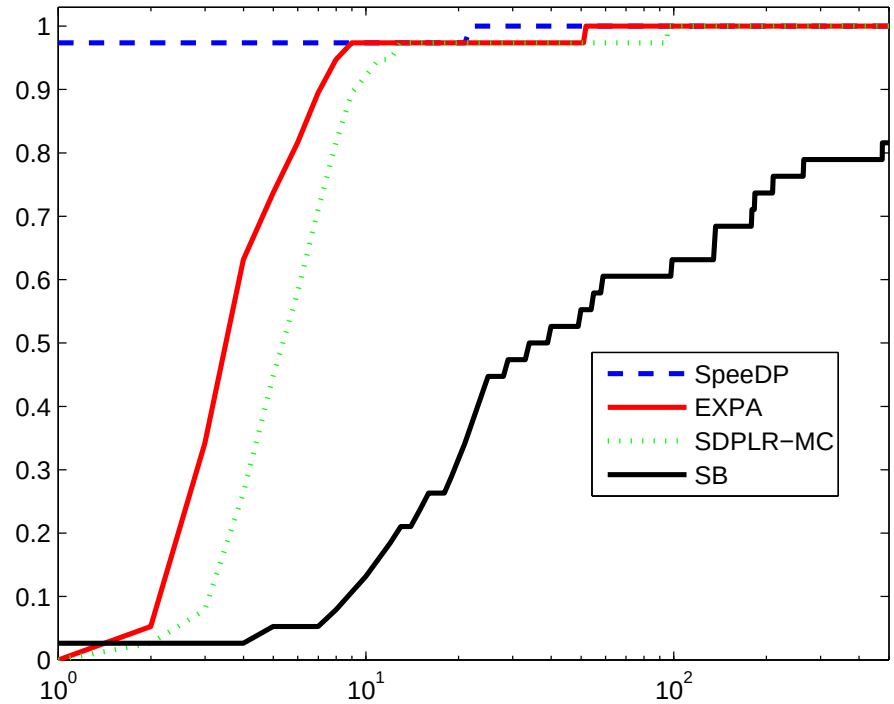

Figure 2: Comparison among the NLP based methods, and SB

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- On the 10 problems where SpeeDP is less accurate the difference is less than 10^{-4}

Some considerations

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  4. The quantity $e^T\lambda + n\lambda_{\min}(Q+\Lambda)$ (and we have computed $\lambda_{\min}(Q+\Lambda)$ to check the optimality condition) is a valid bound for the original SDP relaxation

GW algorithm

- The basic SDP relaxation of Max Cut is at the basis of the randomized algorithm proposed by Goemans and Williamson [Goemans and Williamson, 95] .

- Given the solution of

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- If all the weights of the graph are nonnegative, the expected value of the produced cut is guaranteed to be greater than 0.87856 times the optimal value of max cut
- This heuristic is very used in practice, also as starting point for other heuristics (for example as in BiqMac).

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- It gets for free the vectors v_i needed by the GW algorithm
- The time can be even reduced by tuning the accuracy: we can solve the problem (NLP_r) for a small r and then get the bound $e^T\lambda + \lambda_{\min}(Q + \Lambda)$ (that again we have for free)

A fast heuristic for large graphs II

Consider again problem

$$\begin{array}{ll} \min & \langle Q, X \rangle \\ & \mathrm{diag}(X) = e \\ & X \succeq 0, X \in \mathcal{S}^n \end{array} \qquad (\mathrm{SDP_{MC}})$$

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However, a factorization of X' is needed $\Rightarrow$ impractical for large graphs

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Then the GW rounding and the 1-opt improvement are repeated as well. The whole procedure is repeated a few times with different values of β.

SpeeDP-MC

ALGORITHM SpeeDP-MC

Data: $Q,\ \hat{x} = e,\ \alpha > 0,\ k_{\max},\ \overline{Q} = \sum_{i,j} |Q_{ij}|/|E|.$

For $k = k_{max}, \ldots, 0$ **do** :

S.0 Set $\beta = k\alpha\overline{Q}$ and $Q' = Q + \beta(\hat{x}\hat{x}^T)$

S.1 Apply SpeeDP to problem $(\mathrm{SDP_{MC}})$ with $Q = Q'$ and let v_i, $i = 1, \ldots, n$ be the returned solution and the valid bound ϕ on the max cut problem with objective function corresponding to Q'.

S.2 Apply the Goemans-Williamson hyperplane rounding technique to the vectors v_i, $i = 1, \ldots, n$. This gives a bipartition representative vector $\bar{x}$.

S.3 Apply the 1-opt improvement to $\bar{x}$. This gives a new bipartition representative vector $\tilde{x}$. If $\langle Q, \tilde{x}\tilde{x}^T\rangle < \langle Q, \hat{x}\hat{x}^T\rangle$, set $\hat{x} = \tilde{x}$.

Return Best cut $\hat{x}$, lower bound $\langle -Q, \hat{x}\hat{x}^T\rangle$, upper bound ϕ.

Some considerations

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- The algorithm provides both a lower and upper bound, giving a performance guarantee

Numerical results

We used the graph generator *rudy* to define instances with growing dimension and density and different weights.

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We first considered graphs with number of nodes n equal to $500 + i \cdot 250$, for $i = 0, \ldots, 8$ and with edge $density$ equal to $10\% + i \cdot 10\%$ for $i = 0, \ldots, 9$. For each pair $(n, density)$ we generated three different graphs with positive weights ranging between 1 and 100.

Some numbers

	cut	time	gap %	0.87856*ub
n=500				
	388004.6667	6.216666667	4.595666667	356550.8369
	730297.3333	8.366666667	3.421133333	663560.8102
	1063049	13.4	2.761066667	959739.3467
	1391196.333	18.62666667	2.394766667	1251519.1163
	1714089	19.37666633	1.992666667	1535937.3951
	2032992.333	36.21333233	1.752933333	1817416.2683
	2347809	27.92000033	1.5559	2094784.6522
	2661106	30.53333433	1.295366667	2368226.2755
	2971604.333	44.373333	1.0805	2638941.8736
	3279198.333	47.47666667	0.880966667	2906354.3058
n=1500				
	3220711	60.03666533	3.394633333	2925643.823
	6195972	203.75	2.443333333	5576536.602
	9128485.333	305.3333283	1.932633333	8174914.701
	12020693.33	267.9600067	1.6445	10734573.2
	14899393	338.5299987	1.354266667	13267285.67
	17760397.33	272.16333	2.6834	16022282.32
	20603549.33	347.5099997	2.344366667	18525817.52
	23430844.67	384.0266673	2.016433333	21000489.11
	26236188	341.8233337	1.707533333	23443644.4
	29026448	521.1066593	1.414566667	25862214.06
n=2500				
	8707864.333	326.813334	2.866133333	7869652.293
	16883800	512.99999	2.059233333	15138884.38
	24971508.67	834.2133483	2.199433333	22421516.56
	32984345.33	1475.053324	1.823933333	29507260.52
	40946184	1444.76001	2.447766667	36854243.7
	48873544	1019.210001	1.725333333	43679150.81
	56771676	2992.886719	1.486266667	50618557.71
	64627920	1933.013305	1.558366667	57664353.54
	72453770.67	2257.273397	1.352333333	64515830.64
	80230896	2650.269979	0.905633333	71126099.68

A picture

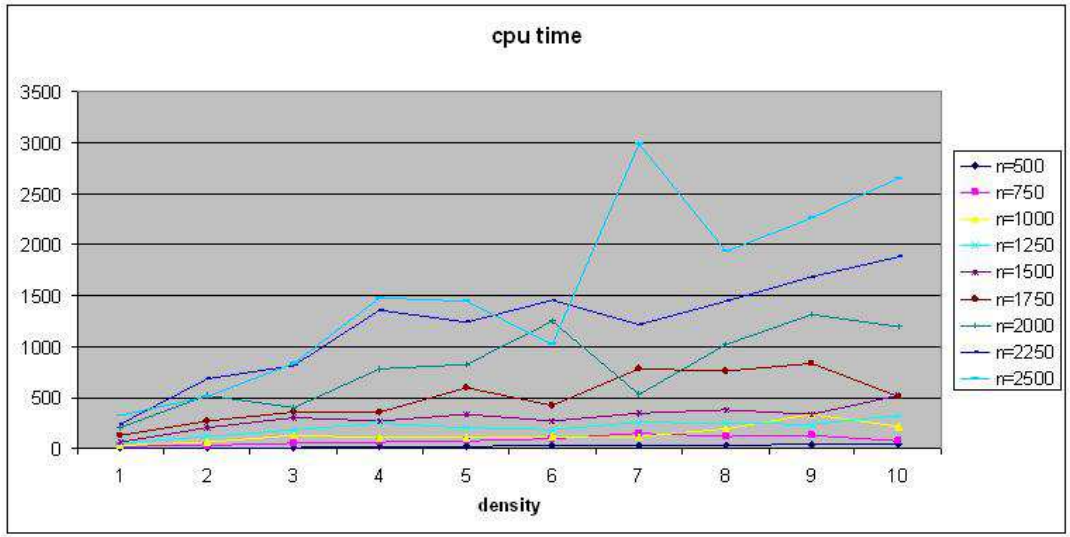

Figure 3: Average CPU time of the heuristic on the random graphs

A picture

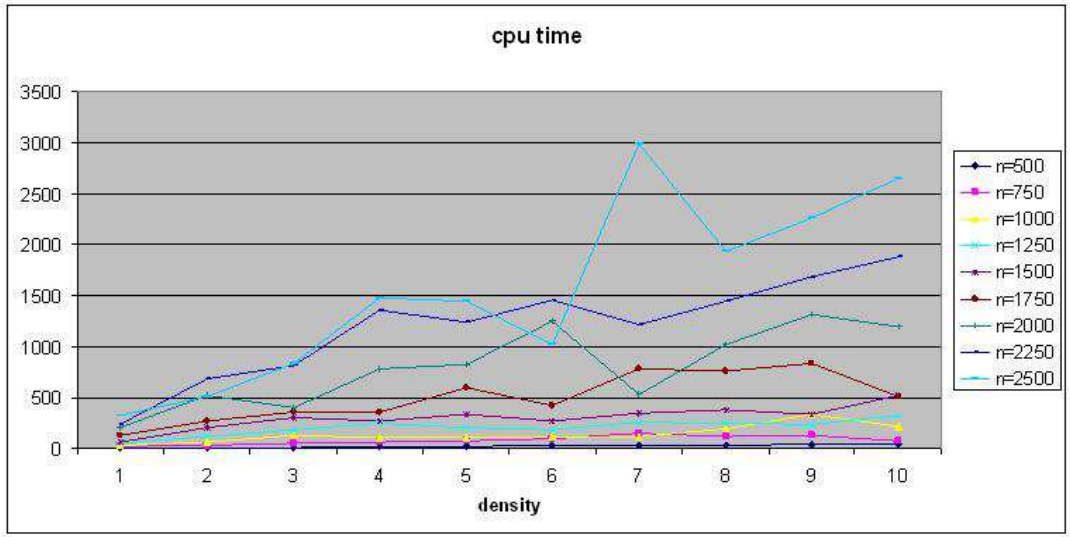

Figure 3: Average CPU time of the heuristic on the random graphs

The heuristic is able to produce a good cut in a small amount of time, and as expected the performance of the heuristic is better on sparse graphs in term of time, but the gap decreases when the density of the graph increases.

Some huge graphs

We generate three random graphs with $100\,001$ nodes, $7\,050\,827$ edges and different weights. The results are in Table 1 where we report the ranges of the weights, the total time, the value of the bound, the best cut obtained and the % gap.

Weights	Total CPU time	Upper Bound	Best Cut	gap%
1	15 043.98	4 113 227.8	3 959 852	3.87
$[1, 100]$	15 142.22	212 076 831.2	203 236 495	4.35
$[-1000, 1000]$	15 919.40	21 006 071 437.9	20 129 935 523	4.35

Table 1: Random sparse graphs with 100 001 nodes and 7 050 827 edges

Some huge graphs

We also generated 6-regular graphs (3D toroidal grid graphs) with 1 030 301 nodes and 3 090 903 edges and different weights. The results are reported in Table 5. To the best of our knowledge, no other methods can achieve this accuracy for graphs of this size.

Weights	Total CPU time	Upper Bound	Best Cut	gap%
1	4 723	3 090 133	3 060 300	0.97
$[1, 10]$	22 042	15 454 739	15 338 007	0.76
$[1, 1000]$	29 072	1 545 550 679	1 534 441 294	0.72
$[-100, 100]$	47 491	57 288 795	49 111 079	14.27

Table 2: 6-regular graphs with 1 030 301 nodes and 3 090 903 edges

Conclusions

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Conclusions

- SpeeDP is a very fast and accurate method for solving the SDP relaxation of Max Cut
- SpeeDP may be useful in order to increase the dimension of the problems solved with an exact algorithm
- SpeeDP allows to define efficient heuristics that produce feasible cuts with a certified upper limit on the distance from optimality
- SpeeDP-MC allows to find very good cuts for graphs with million of edges

Future work

- Extend SpeeDP to the bisection problem

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- Define a parallel version of the code. The main burden in computing the function

$$f_\epsilon(v) := \sum_{i,\ j} q_{ij} \frac{v_i^T v_j}{\|v_i\|\|v_j\|} + \frac{1}{\epsilon} \sum_{i=1}^{n} \frac{(\|v_i\|^2 - 1)^2}{d(v_i)},$$

and its gradient is the vector with the i-th component equal to $\sum_j q_{ij} \frac{v_j}{\|v_j\|}$, but this can be parallelized, getting a code that is faster and able to tackle even larger graphs.