Sums of Squares and Semidefinite Programs: A Tutorial

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Sums of squares of polynomials

Notation:

$\mathcal{P}_{n,d}$ denotes the n-variate polynomials of total degree at most d.

Definition:

A polynomial $p \in \mathcal{P}_{n,2d}$ is called a *sum-of-squares (SOS)* if

$$p(x) = \sum_{i=1}^{k} p_i^2(x)$$

for some polynomials $p_i \in \mathcal{P}_{n,d}$ $(i = 1, \ldots, k)$.

Notation:

The SOS n-variate polynomials are denoted by Σ_n, and $\Sigma_{n,d} := \mathcal{P}_{n,d} \cap \Sigma_n$.

Gram matrix representation

Lemma:

$\mathcal{P}_{n,d}$ has dimension $\binom{n+d}{d}$ (as a vector space).

For example, a basis for $\mathcal{P}_{2,3}$ is

$$1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3.$$

Theorem

One has $p \in \Sigma_{n,2d}$ if and only if

$$p(x) = B_{n,d}(x)^T M B_{n,d}(x),$$

where $B_{n,d}(x)$ is any fixed basis for $\mathcal{P}_{n,d}$, and M is a positive semidefinite matrix of size $\binom{n+d}{d} \times \binom{n+d}{d}$.

Example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

YES, because

$$P(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

The 3×3 matrix (say M) is positive semidefinite and:

$$M = L^TL, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

and consequently, using $\tilde{x} = [x_1^2 \ \ x_2^2 \ \ x_1x_2]^T$,

$$\begin{aligned} p(x) &= \tilde{x}^TM\tilde{x} = \tilde{x}^TL^TL\tilde{x} = \|L\tilde{x}\|^2 \\ &= \frac{1}{2}\left(2x_1^2 - 3x_2^2 + x_1x_2\right)^2 + \frac{1}{2}\left(x_2^2 + 3x_1x_2\right)^2. \end{aligned}$$

SDP reformulations

One can reformulate the question: is $p \in \Sigma_{n,2d}$? as an SDP in two ways, using two principles:

- Two polynomials are equal if all their coefficients are equal ...
- ... or if they have the same function values at "suitably chosen" points.

SDP reformulation I

Notation:

$$\mathbb{N}^n_d := \left\{ \alpha \in \mathbb{N}^n_0 \mid \sum_{i=1}^{n} \alpha_i \le d \right\}.$$

For $\alpha \in \mathbb{N}^n_d$ and $x \in \mathbb{R}^n$:

$$x^\alpha := x_1^{\alpha_1} \times \ldots \times x_n^{\alpha_n}.$$

If $p \in \mathcal{P}_{n,d}$ we may write: $p(x) = \sum_{\alpha \in \mathbb{N}^n_d} p_\alpha x^\alpha$.

SDP formulation

One has $p \in \Sigma_{n,2d}$ if and only if there exists a positive semidefinite matrix M of size $\binom{n+d}{d} \times \binom{n+d}{d}$, such that

$$\sum_{\gamma,\beta \in \mathbb{N}^n_d,\ \gamma+\beta=\alpha} M_{\gamma,\beta} = p_\alpha \quad \forall\ \alpha \in \mathbb{N}^n_{2d}.$$

SDP reformulation II

Lemma

Let $p_1, p_2 \in \mathcal{P}_{n,d}$ *and*

$$\Delta(n,d) := \left\{ x \in \mathbb{R}^n \;:\; dx \in \mathbb{N}_0^n,\; \sum_{i=1}^{n} x_i = 1 \right\}.$$

One has $p_1 = p_2$ *iff*

$$p_1(x) = p_2(x) \quad \forall\, x \in \Delta(n,d).$$

SDP formulation

One has $p \in \Sigma_{n,2d}$ if and only if

$$p(x) = B_{n,d}(x)^T M B_{n,d}(x) \quad \forall\, x \in \Delta(n,d),$$

where $B_{n,d}(x)$ is any fixed basis for $\mathcal{P}_{n,d}$, and M is a positive semidefinite matrix of size $\binom{n+d}{d} \times \binom{n+d}{d}$.

Software: SOSTools

The SDP approach to sum-of square-decompositions is implemented in the free Matlab software *SOSTools*.

S. Prajna and A. Papachristodoulou and P. Seiler and P. A. Parrilo, SOSTOOLS: Sum of squares optimization toolbox for MATLAB, Available from `http://www.cds.caltech.edu/sostools`, 2004.

SOSTools requires an SDP solver, e.g. *SeDuMi*.

SOSTools code for Parrilo example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$ a sum of squares?

```
% Introduce the variables
mpvar(2,1,'x');

% Define the polynomial
p = 2*x(1)^4 + 2*x(1)^3*x(2) - x(1)^2*x(2)^2 + 5*x(2)^4;

% Test if p is SOS
[Q,Z]=findsos(p)
```

SOSTools output in Matlab

```
Q =

   5.000000000000336   0.000000000000001  -1.678812742759978
   0.000000000000001   2.357625485522271   0.999999999999674
  -1.678812742759978   0.999999999999674   2.000000000001323


Z =
  [      x_2_1^2 ]
  [ x_1_1*x_2_1 ]
  [      x_1_1^2 ]
```

One may verify that $Q \succeq 0$ and $p(x) \approx Z^T Q Z = \|Q^{\frac{1}{2}} Z\|^2$. Note that Q is different from before! (Not unique).

SOS vs nonnegativity

A nonnegative polynomial is not necessarily a sum of squares of polynomials (SOS).

Example (Motzkin):

The form

$$M(x,y,z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$$

is nonnegative, since

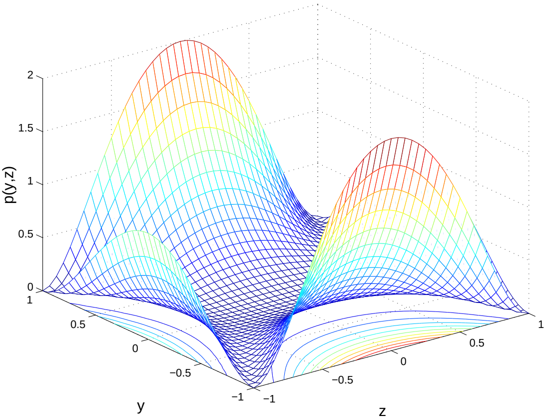
$$\begin{aligned} M(x,y,z) &= \left(\frac{(x^2-y^2)\,z^3}{(x^2+y^2)}\right)^2 + \left(\frac{x^2y\,(x^2+y^2-2z^2)}{(x^2+y^2)}\right)^2 \\ &+ \left(\frac{xyz\,(x^2+y^2-2z^2)}{(x^2+y^2)}\right)^2, \end{aligned}$$

but M is not an SOS.

SOS vs nonnegativity

Motzkin example with $x = 1$:

$$p(y,z) = z^6 + y^2 + y^4 - 3z^2y^2.$$



SOS vs nonnegativity (ctd)

Theorem (Hilbert)

Nonnegativity and *sum of squares* are the same for polynomials of degree d on n variables in precisely the following cases:

- $n = 1$ (univariate polynomials);
- $d = 2$ (quadratic polynomials on n variables);
- $n = 2$ and $d \leq 4$ (bivariate polynomials of degree at most 4).

Computational complexity

To check nonnegativity of a $p \in \mathcal{P}_{n,d}$ is NP-hard if $d \geq 4$.

Artin's theorem

Artin's theorem (Hilbert's 17th problem):

Let $p : \mathbb{R}^n \mapsto \mathbb{R}$ be a multivariate polynomial. Then $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$ iff

$$p \sum_j q_j^2 = \sum_i p_i^2$$

for some polynomials p_i and q_j.

Implication:

One may obtain a *certificate* of nonnegativity of p via semidefinite programming. (Adapt the Gram matrix method.)

Unconstrained optimization: $p^* := \min_{x \in \mathbb{R}^n} p(x)$

If $p \in \mathcal{P}_{n,d}$ then:

$$\begin{aligned} p^* &= \sup\{\rho \;:\; p(x) - \rho \geq 0 \;\; \forall x \in \mathbb{R}^n\} \\ &\geq \sup\{\rho \;:\; p(x) - \rho \in \Sigma_{n,d} \;\; \forall x \in \mathbb{R}^n\}. \end{aligned}$$

- Thus we may compute an SDP lower bound on p^*.
- This may be done using SOSTools.

Further reading

P.A. Parrilo, B. Sturmfels, Minimizing polynomial functions. In *Algorithmic and quantitative real algebraic geometry*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 60, pp. 83–99, AMS, 2003.

Putinar's theorem

Consider *compact semi-algebraic set*

$$S = \{x \in \mathbb{R}^n \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, k)\}.$$

Assumption:

There exists a

$$\bar{p} \in \Sigma_n + p_1\Sigma_n + \ldots + p_k\Sigma_n$$

such that $\{x \ : \ \bar{p}(x) \geq 0\}$ is *compact*.

Theorem (Putinar):

For a given polynomial p_0 one has $p_0(x) > 0$ for all $x \in S$ iff

$$p_0 \in \Sigma_n + p_1\Sigma_n + \ldots + p_k\Sigma_n.$$

M. Putinar. Positive polynomials on compact semi-algebraic sets. *Ind. Univ. Math. J.* 42:969–984, 1993.

Lasserre's approach

Consider the minimization problem

$$p^* = \min_{x \in S} p(x).$$

By Putinar's theorem we have

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} p(x) \\ = \quad & \sup \{\rho \ : \ p(x) - \rho > 0 \ \ \forall x \in S\} \\ = \quad & \sup \{\rho \ : \ (p - \rho) \in \Sigma_n + p_1 \Sigma_n + \ldots + p_k \Sigma_n\} \\ \geq \quad & \sup \{\rho \ : \ (p - \rho) \in \Sigma_{n,t} + p_1 \Sigma_{n,t} + \ldots + p_k \Sigma_{n,t}\} \\ := \quad & \rho_t \ \ (\text{for any integer } t \geq 1). \end{aligned}$$

We have that $\rho_i \leq \rho_{i+1} \leq p^*$ and

$$\lim_{t \to \infty} \rho_t = p^*.$$

J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIOPT*, 11:296–817, 2001.

Lasserre's approach (ctd)

Return to the unconstrained optimization problem:

$$p^* = \min_{x \in \mathbb{R}^n} p(x).$$

Artificial constraint $\|x\|^2 \leq R$ for some 'sufficiently large' R.

Now we have $\min_{x \in S} p(x)$ where S is the compact semi-algebraic set

$$S := \left\{ x \in \mathbb{R}^n \ : \ R - \|x\|^2 \geq 0 \right\}.$$

No a priori choice for R available in general.

Software: Gloptipoly

Lasserre'a approach is implemented in the software *Gloptipoly*.

D. Henrion, J. B. Lasserre, J. Loefberg. GloptiPoly 3: moments, optimization and semidefinite programming. *Optimization Methods and Software*, **24**:4-5, 761–779, 2009.

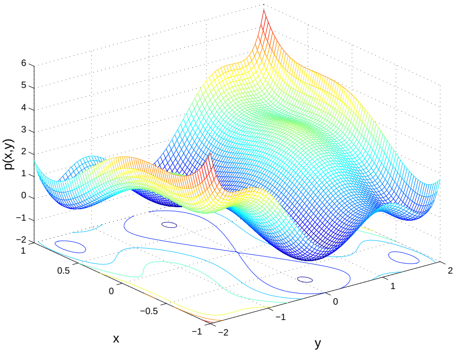
Gloptipoly requires an SDP solver, e.g. *SeDuMi*. Freely available at:

`http://homepages.laas.fr/henrion/software/gloptipoly3/`

GloptiPoly extremely useful to prove *global optimality* in small problems.

Example

$$\min p(x,y) := x^2(4 - 2.1x^2 + \frac{1}{3}x^4) + xy + y^2(-4 + 4y^2);$$



Six local minima. Two global minima: (0.0898 -0.7127) and (-0.0898 0.7127).
Gloptipoly demonstration follows ...

Gloptipoly code

```
% First we define the variables
% and the polynomial to be minimized

mpol x1 x2
g0 = 4*x1^2+x1*x2-4*x2^2-2.1*x1^4+4*x2^4+x1^6/3

% Then we define the optimization problem

P = msdp(min(g0))

% Solve the SDP formulation using e.g. Sedumi

[status,obj] = msol(P)

% Display results
obj
x = double([x1 x2])
```

Gloptipoly output in Matlab

```
obj =

  -1.031628452481396


x(:,:,1) =

   0.089847645153081  -0.712650754620992


x(:,:,2) =

  -0.089847645153052   0.712650754621033
```

Thus both global minimizers are found.

The End

Further reading:

M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In *Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds.), Springer, pages 157-270, 2009

`http://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf`

THANK YOU!