Exercises

1. Any principal submatrix of a positive semidefinite matrix is again positive semidefinite.

2. If a diagonal element of a positive semidefinite matrix is zero, so are all offdiagonal elements of this row.

3. For any given regular $B \in \mathbb{R}^{n\times n}$ there holds $A \in \mathcal{S}^n_+ \Leftrightarrow BAB^T \in \mathcal{S}^n_+$.

4. Given orthogonal $P \in \mathbb{R}^{n\times k}$ ($P^TP = I_k$) and $A \in \mathcal{S}^n_+$, assume $A \notin \{PUP^T : U \succeq 0\}$. Then there is a $v \in \mathbb{R}^n$ with $P^Tv = 0$ and $v^TAv > 0$.

5. $A \in \mathcal{S}^n_+ \qquad \Leftrightarrow \qquad \langle A, B\rangle \geq 0 \quad \forall B \in \mathcal{S}^n_+$.

6. (*) Let $P \in \mathbb{R}^{n\times k}$ be an orthogonal matrix ($P^TP = I_k$), then $\{PUP^T : U \in \mathcal{S}^k_+\}$ is a face of $\mathcal{S}^n_+$ and any face of $\mathcal{S}^n_+$ of dimension at least one has such a representation.

7. Semidefinite Scaling: Given regular $B \in \mathbb{R}^{n\times n}$, the semidefinite programs

$$\inf\{\langle C, X\rangle : \langle A_i, X\rangle = b_i, i = 1, \ldots, m, X \succeq 0\}$$

and

$$\inf\{\langle BCB^T, \overline{X}\rangle : \langle BA_iB^T, \overline{X}\rangle = b_i, i = 1, \ldots, m, \overline{X} \succeq 0\}$$

have the same optimal value and there is an automorphism on S^n_+ that bijectively maps feasible/optimal X-solutions to feasible/optimal $\overline{X}$-solutions.

8. (*) Show that the elliptope relaxation

$$\min\{\langle C, X\rangle : X_{ii} = 1, i = 1, \ldots, n, X \succeq 0\}$$

and the quadratic 0-1 relaxation

$$\min\{\langle Q, \overline{X}\rangle : \overline{X}_{11} = 1, \overline{X}_{1i} = \overline{X}_{ii}, i = 2, \ldots, n, \overline{X} \succeq 0\}$$

are scalings of each other.

9. (**) Prove or disprove: If $\{X \succeq 0 : \langle A_i, X\rangle = b_i, i = 1, \ldots, m\}$ is bounded, there is a scaling by some regular $B \in \mathbb{R}^n \times n$ so that for matrices in $\{\overline{X} = BXB^T \succeq 0 : \langle B^{-T}A_iB^{-1}, \overline{X}\rangle = b_i, i = 1, \ldots, m\}$ the trace is constant.

10. (*) Let $\hat{X} \in \mathcal{S}^n_+$ be a point in the relative interior of the feasible set of $\inf\{\langle C, X\rangle : \langle A_i, X\rangle = b_i, i = 1, \ldots, m, X \succeq 0\}$ having eigenvalue decompostions $\hat{X} = P\Lambda P^T$ with diagonal positive definite $\Lambda \in \mathcal{S}^k_+$, $k \leq n$ and $P^TP = I_k$. Prove that the program $\inf\{\langle PCP^T, U\rangle : \langle PA_iP^T, U\rangle = b_i, i = 1, \ldots, m, U \succeq 0\}$ is equivalent to the first and that its dual satisfies strong duality.

11. Let $\hat{M} \subset \mathbb{R}^{m+1}$ be some convex compact set and $\hat{y} \in \mathbb{R}^m$. Find a saddle point theorem that proves the existence of saddlepoints for the problem

$$\inf_{y \in \mathbb{R}^m} \sup_{(\gamma,g)\in\hat{M}, \eta\in\mathbb{R}^m_+} [\gamma + \langle g - \eta, y\rangle + \tfrac{1}{2}\|y - \hat{y}\|^2].$$

12. (*) Extend the Gauss-Seidel approach for dealing with sign constraint variables to box-constraints.

13. (**) Extend the spectral bundle approach for SDP to second order cone problems. Show that in this case there is no need for a residual part in the model and a P with at most three columns suffices to guarantee convergence.

14. (*) Try to complete the proof on the bound on the number of strong nodal domains, i.e., in the notation of the sketch, show $g^T M g \leq \lambda_k$ (this might need an idea).

15. (*) Given a connected graph $G = (N, E)$, prove that the second smallest eigenvalue of the Laplace matrix of G is strictly positive (hint: use Perron-Frobenius or the Matrix Tree Theorem).

16. (*) Given a connected graph $G = (N, E)$, denote the weighted Laplacian by $L_w(G)$ and let $\hat{a}(G) := \sup\{\lambda_2(L_w(G)) : \sum_{ij\in E} w_{ij} = |E|, w \in \mathbb{R}^E_+\}$. Prove

$$\begin{array}{rll} \frac{|E|}{\hat{a}(G)} = \text{maximize} & \sum_{i\in N} \|v_i\|^2 & \\ \text{subject to} & (\sum_{i\in N} v_i)^2 = 0, & \\ & \|v_i - v_j\|^2 \leq 1 & \text{for } ij \in E, \\ & v_i \in \mathbb{R}^n \text{ for } i \in N. & \end{array}$$

In proving this, make certain that strong duality holds whenever you dualize!

17. (*) Given a tree $G = (N, E)$, let v_i, $i = 1, \ldots, n$ be an optimal embedding corresponding to the maximized second smallest eigenvalue of the Laplace matrix of G. Show that the originr either coincides with a v_i for some $i \in N$ or there is an edge $ij \in E$ with $0 \in [v_i, v_j]$.