Graph Realizations Corresponding to Optimized Extremal Eigenvalues of the Laplacian

Christoph Helmberg

joint work with

Frank Göring

Susanna Reiß (Dienelt)

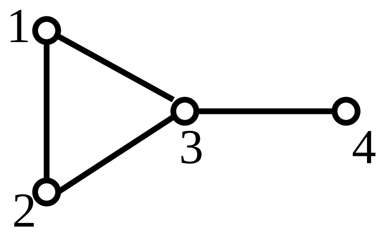
Markus Wappler

TU Chemnitz

- Introduction: Laplacian and Algebraic Connectivity
- Eigenvalue Optimization and Embedding Problems
- Separators and Optimal Embeddings
- Tree-Width Bound on Minimal Dimensions
- Sharpness of the Bounds
- Rotational Dimension of a Graph

Introduction

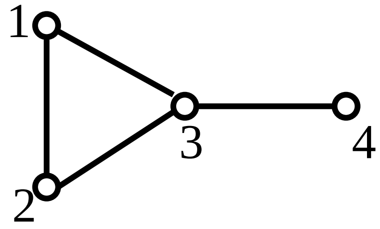
- simple undirected Graph $G = (N, E)$, $N = \{1, \ldots, n\}$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \text{Diag}(Ae) - A$ $\quad A$ ...adjacency matrix, $e = [1, 1, \ldots, 1]^T$



$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Introduction

- simple undirected Graph $G = (N, E)$, $N = \{1, \ldots, n\}$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \text{Diag}(Ae) - A$ $\quad A \ldots$ adjacency matrix, $e = [1, 1, \ldots, 1]^T$

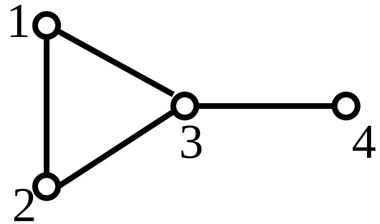


$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- weighted Laplacian for $w \geq 0$: $L_w(G) = \sum_{ij \in E} w_{ij} E_{ij}$ $\qquad E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$

Introduction

- simple undirected Graph $G = (N, E)$, $N = \{1, \ldots, n\}$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \text{Diag}(Ae) - A$ $\quad A$ . . . adjacency matrix, $e = [1, 1, \ldots, 1]^T$

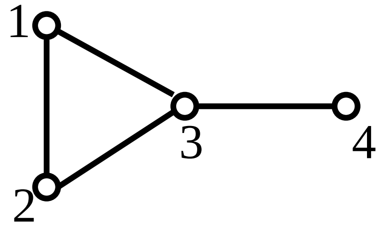


$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- weighted Laplacian for $w \geq 0$: $L_w(G) = \sum_{ij \in E} w_{ij} E_{ij}$ $\qquad E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$
- Properties: $\quad L_w \succeq 0$ (sym., pos. semidef.), $\qquad \lambda_1(L_w) = 0$ with EV e.

Introduction

- simple undirected Graph $G = (N, E)$, $N = \{1, \ldots, n\}$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \text{Diag}(Ae) - A$ $\quad A \ldots$ adjacency matrix, $e = [1, 1, \ldots, 1]^T$



$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- weighted Laplacian for $w \geq 0$: $L_w(G) = \sum_{ij \in E} w_{ij} E_{ij}$ $\qquad E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$
- Properties: $L_w \succeq 0$ (sym., pos. semidef.), $\qquad \lambda_1(L_w) = 0$ with EV e.

- [Fiedler 1973, 1989, 1993] $\qquad \lambda_2(L(G)) > 0$ iff G is connected
  proved close ties to edge and node connectivity, "algebraic connectivity"
- EV to $\lambda_2(L)$ used in many partitioning heuristics, "Fiedler vector"
- Laplacian Spectrum in Graph Theory, see e.g. [Mohar 1991, 2004]

Central Question:

What connections exist between eigenvectors of extremal eigenvalues and structural properties of the graph?

Central Question:

What connections exist between eigenvectors of extremal eigenvalues and structural properties of the graph?

Idea: redistribute the weight on the edges of the graph, then hopefully characteristic properties will become more apparent.

Central Question:

What connections exist between eigenvectors of extremal eigenvalues and structural properties of the graph?

Idea: redistribute the weight on the edges of the graph, then hopefully characteristic properties will become more apparent.

[Fiedler 1989] did this for λ_2

$$\max_{w \in \mathcal{W}} \; \lambda_2(L_w) \qquad\qquad \mathcal{W} = \{w \in \mathbb{R}^E_+ : \sum_{ij \in E} w_{ij} = w^T e \leq 1\}$$

→ *Absolute Algebraic Connectivity*

[Fiedler 1990] exhibits formula for optimal weights in trees.

From Eigenvalues to Embeddings in the Eigenspace

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$

From Eigenvalues to Embeddings in the Eigenspace

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
as SDP:	
max λ s.t. $\sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq \lambda I$ $e^T w = 1, w \geq 0$	min λ s.t. $\sum_{ij \in E} w_{ij} E_{ij} \preceq \lambda I$ $e^T w = 1, w \geq 0$

From Eigenvalues to Embeddings in the Eigenspace

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
as SDP:	
$\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq \lambda I \\ & e^T w = 1, w \geq 0 \end{array}$	$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} \preceq \lambda I \\ & e^T w = 1, w \geq 0 \end{array}$
divide by $\lambda_{opt} > 0$	
$\begin{array}{ll} \min & e^T w \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq I \\ & w \geq 0 \end{array}$	$\begin{array}{ll} \max & e^T w \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} \preceq I \\ & w \geq 0 \end{array}$

From Eigenvalues to Embeddings in the Eigenspace

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
as SDP:	
max λ s.t. $\sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq \lambda I$ $e^T w = 1, w \geq 0$	min λ s.t. $\sum_{ij \in E} w_{ij} E_{ij} \preceq \lambda I$ $e^T w = 1, w \geq 0$
divide by $\lambda_{opt} > 0$	
min $e^T w$ s.t. $\sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq I$ $w \geq 0$	max $e^T w$ s.t. $\sum_{ij \in E} w_{ij} E_{ij} \preceq I$ $w \geq 0$
dualize	
max $\langle I, X \rangle$ s.t. $\langle E_{ij}, X \rangle \leq 1, \quad ij \in E$ $\langle e e^T, X \rangle = 0$ $X \succeq 0$	min $\langle I, X \rangle$ s.t. $\langle E_{ij}, X \rangle \geq 1, \quad ij \in E$ $X \succeq 0$

From Eigenvalues to Embeddings in the Eigenspace

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$

as SDP:

$\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq \lambda I \\ & e^T w = 1, w \geq 0 \end{array}$	$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} \preceq \lambda I \\ & e^T w = 1, w \geq 0 \end{array}$

divide by $\lambda_{opt} > 0$

$\begin{array}{ll} \min & e^T w \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq I \\ & w \geq 0 \end{array}$	$\begin{array}{ll} \max & e^T w \\ \text{s.t.} & \sum_{ij \in E} w_{ij} E_{ij} \preceq I \\ & w \geq 0 \end{array}$

dualize

$\begin{array}{ll} \max & \langle I, X \rangle \\ \text{s.t.} & \langle E_{ij}, X \rangle \leq 1, \quad ij \in E \\ & \langle e e^T, X \rangle = 0 \\ & X \succeq 0 \end{array}$	$\begin{array}{ll} \min & \langle I, X \rangle \\ \text{s.t.} & \langle E_{ij}, X \rangle \geq 1, \quad ij \in E \\ & X \succeq 0 \end{array}$

embedding: set $X = V^T V$ with $V = [v_1, \ldots, v_n]$

$\begin{array}{ll} \max & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \leq 1, \quad ij \in E \\ & \sum v_i = 0 \\ & v_i \in \mathbb{R}^n \end{array}$	$\begin{array}{ll} \min & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \geq 1, \quad ij \in E \\ & v_i \in \mathbb{R}^n \end{array}$

($\sum v_i = 0$ holds, as well)

The embedding problem

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
$\begin{array}{ll} \max & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \leq 1, \quad ij \in E \\ & \sum v_i = 0 \\ & v_i \in \mathbb{R}^n \end{array}$	$\begin{array}{ll} \min & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \geq 1, \quad ij \in E \\ & (\sum v_i = 0) \\ & v_i \in \mathbb{R}^n \end{array}$

- λ_2: nodes are pulled outwards and edges are cables of length 1
  $\lambda_{\max}$: nodes are pressed inwards and edges are struts of length 1
  w_{ij} is the force acting on the cable/strut $ij \in E$
- For any $u \in \mathbb{R}^n$ the projection $V^T u$ yields an eigenvector to λ_{opt}.
  (by complementarity)

The embedding problem

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
$\begin{array}{ll} \max & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \le 1, \quad ij \in E \\ & \sum v_i = 0 \\ & v_i \in \mathbb{R}^n \end{array}$	$\begin{array}{ll} \min & \sum \|v_i\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \ge 1, \quad ij \in E \\ & (\sum v_i = 0) \\ & v_i \in \mathbb{R}^n \end{array}$

- λ_2: nodes are pulled outwards and edges are cables of length 1
  $\lambda_{\max}$: nodes are pressed inwards and edges are struts of length 1
  w_{ij} is the force acting on the cable/strut $ij \in E$
- For any $u \in \mathbb{R}^n$ the projection $V^T u$ yields an eigenvector to λ_{opt}.
  (by complementarity)
- Replace the origin by the barycenter $\bar{v} = \frac{1}{n} \sum v_i$

$\begin{array}{ll} \max & \sum \|v_i - \bar{v}\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \le 1, \quad ij \in E \\ & \sum (v_i - \bar{v}) = 0 \\ & v_i \in \mathbb{R}^n \end{array}$	$\begin{array}{ll} \min & \sum \|v_i - \bar{v}\|^2 \\ \text{s.t.} & \|v_i - v_j\|^2 \ge 1, \quad ij \in E \\ & \sum (v_i - \bar{v}) = 0 \\ & v_i \in \mathbb{R}^n \end{array}$

then $\quad \sum \|v_i - \bar{v}\|^2 = \frac{1}{n} \sum_{i,j} \|v_i - v_j\|^2 \quad$ is the *variance*.

Equivalent/Related Problems

- λ_2: Fastest Mixing Markov Processes [SunBoydXiaoDiaconis2006]
  λ_2 is the rate of convergence to the stationary distribution
  also give an interpretation as maximizing conductance

- Maximum Variance Unfolding (visualization in data mining) [WeinbergerSaul2004]

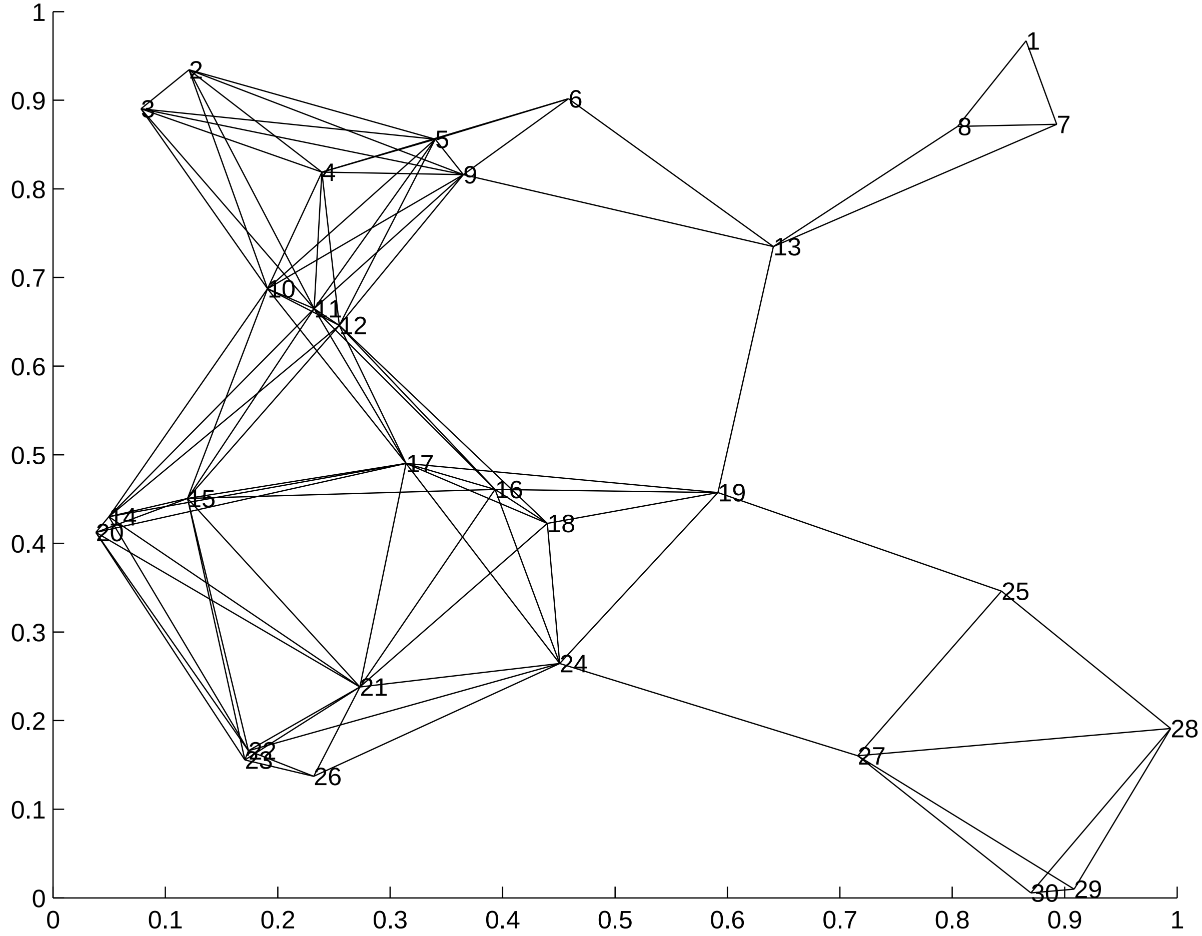
- Tensegrity Theory [Connelly1999]
  the variance corresponds to the potential and L_w to the stress matrix

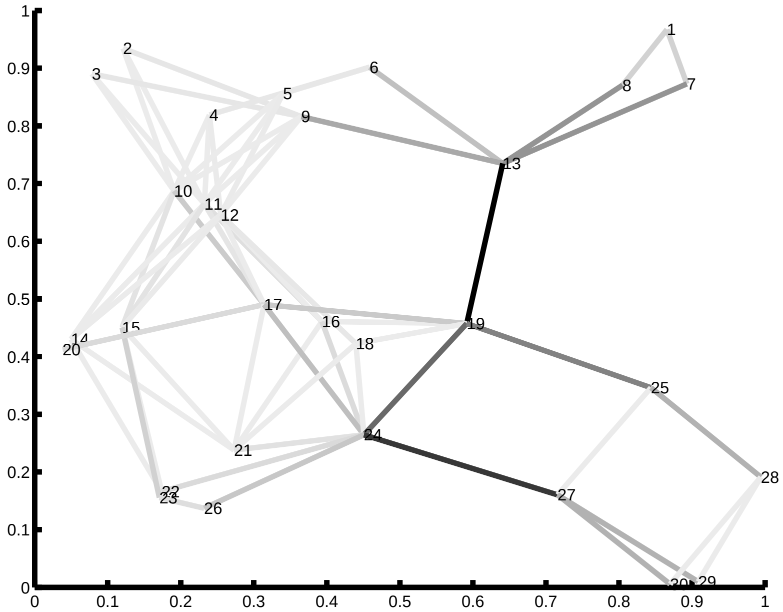
- Graph Realizations [BelkConnelly2007]

- low dimensional embeddings [LinialLondonRabinovich1995]

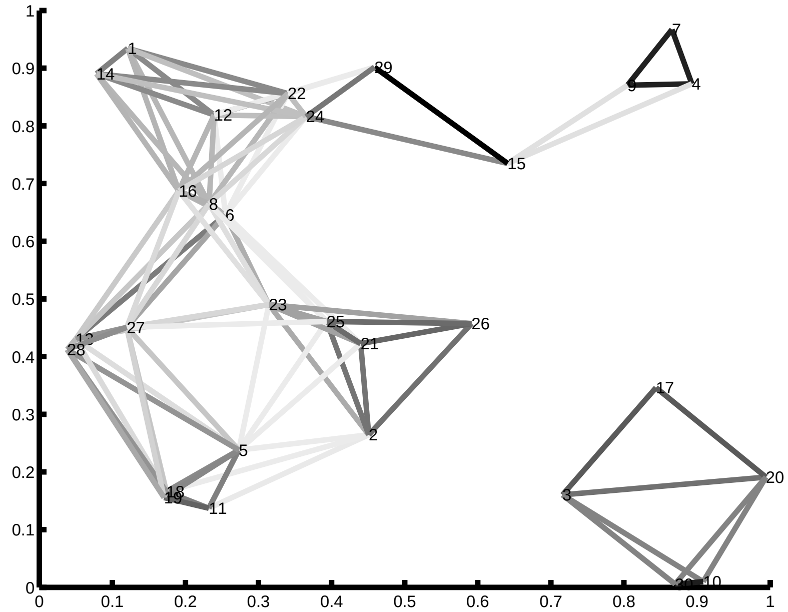
- Expanders [HooryLinialWigderson2006]

- Colin de Verdière and related graph parameters [CdV98,vdHolst03]

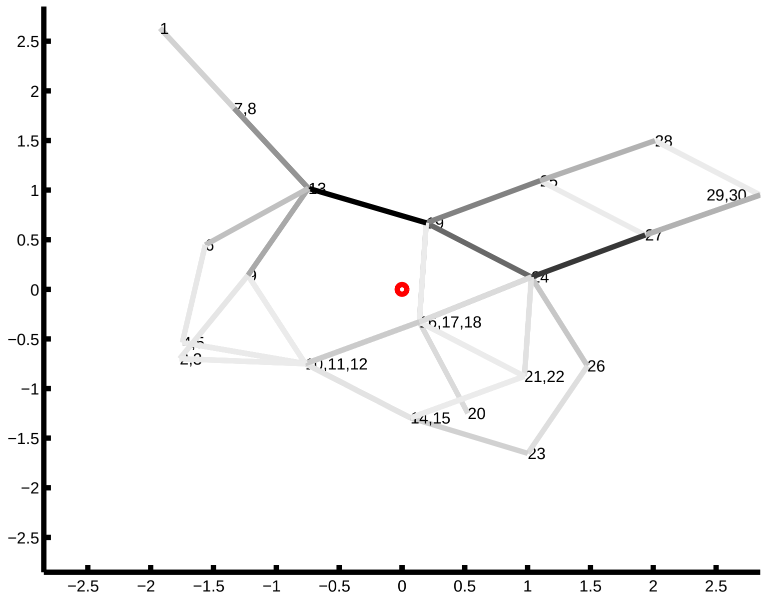

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



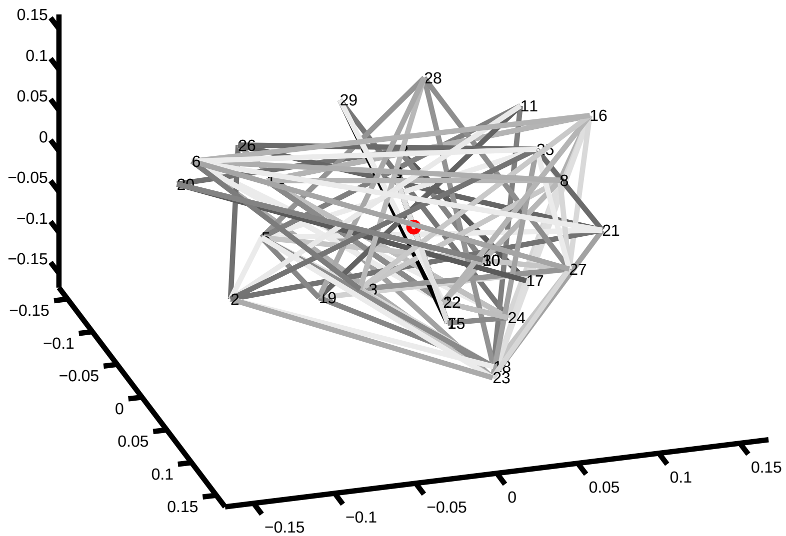
optimal weights for λ_2



optimal weights for λ_{max}



an optimal λ_2-embedding (2D!)



an optimal λ_{max}-embedding (14D?)

Connections to the (node-)separator structure

For a connected graph $G = (N, E)$ the removal of a *(node-)separator* $S \subset N$ disconnects the graph into at least two connected components.

Connections to the (node-)separator structure

For a connected graph $G = (N, E)$ the removal of a *(node-)separator* $S \subset N$ disconnects the graph into at least two connected components.

Tree-Width

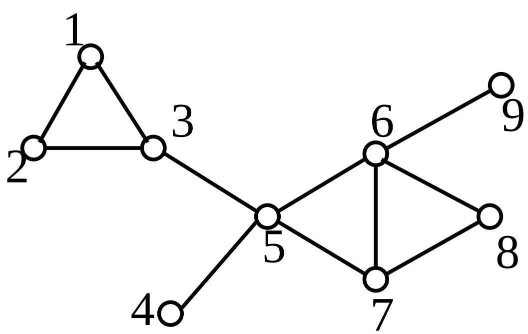
Given $G = (N, E)$, let $T = (\mathcal{N}, \mathcal{E})$ be a tree with $\mathcal{N} \subseteq 2^N$ and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ so that

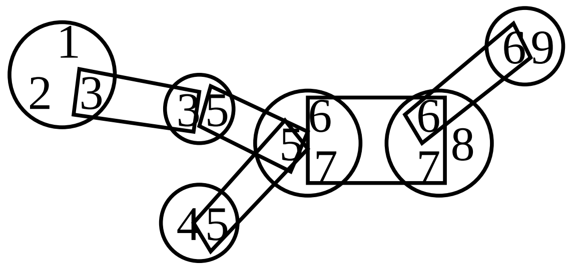
(i) $N = \bigcup_{U \in \mathcal{N}} U$.

(ii) For every $e \in E$ there is a $U \in \mathcal{N}$ with $e \subseteq U$.

(iii) If $U_1, U_2, U_3 \in \mathcal{N}$ with U_2 on the $T-$path from U_1 to U_3, then $U_1 \cap U_3 \subseteq U_2$.

Then T is called a *tree-decomposition* of G.
The *width* of T is the number $\max\{|U| - 1 : U \in \mathcal{N}\}$.
The *tree-width* $tw(G)$ is the least width of any tree-decomposition of G.





Connections to the (node-)separator structure

For a connected graph $G = (N, E)$ the removal of a *(node-)separator* $S \subset N$ disconnects the graph into at least two connected components.

Tree-Width

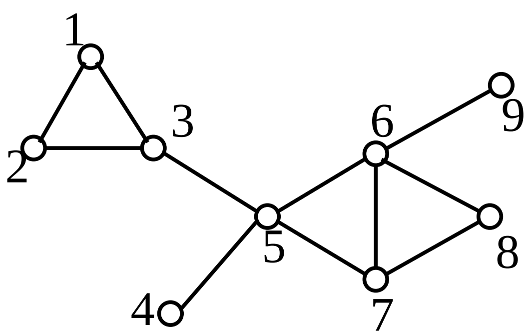
Given $G = (N, E)$, let $T = (\mathcal{N}, \mathcal{E})$ be a tree with $\mathcal{N} \subseteq 2^N$ and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ so that

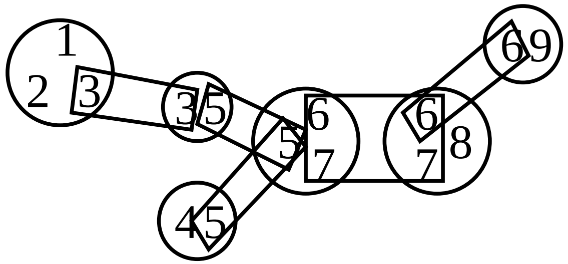
(i) $N = \bigcup_{U \in \mathcal{N}} U$.

(ii) For every $e \in E$ there is a $U \in \mathcal{N}$ with $e \subseteq U$.

(iii) If $U_1, U_2, U_3 \in \mathcal{N}$ with U_2 on the $T-$path from U_1 to U_3, then $U_1 \cap U_3 \subseteq U_2$.

Then T is called a *tree-decomposition* of G.
The *width* of T is the number $\max\{|U| - 1 : U \in \mathcal{N}\}$.
The *tree-width* $tw(G)$ is the least width of any tree-decomposition of G.

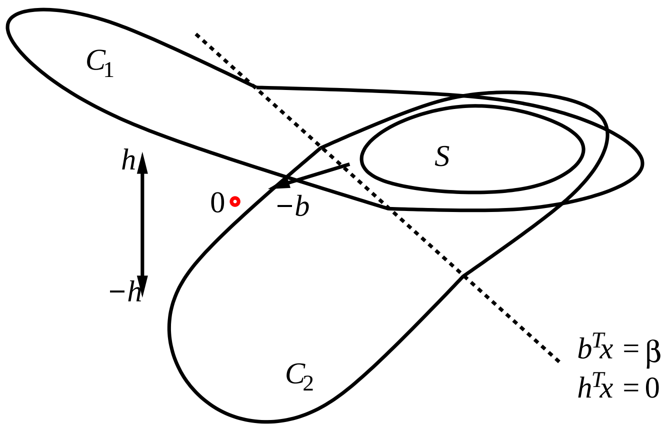




Any $U \in \mathcal{N}$ and any $U \cap U'$ with $\{U, U'\} \in \mathcal{E}$ is a separator of G.

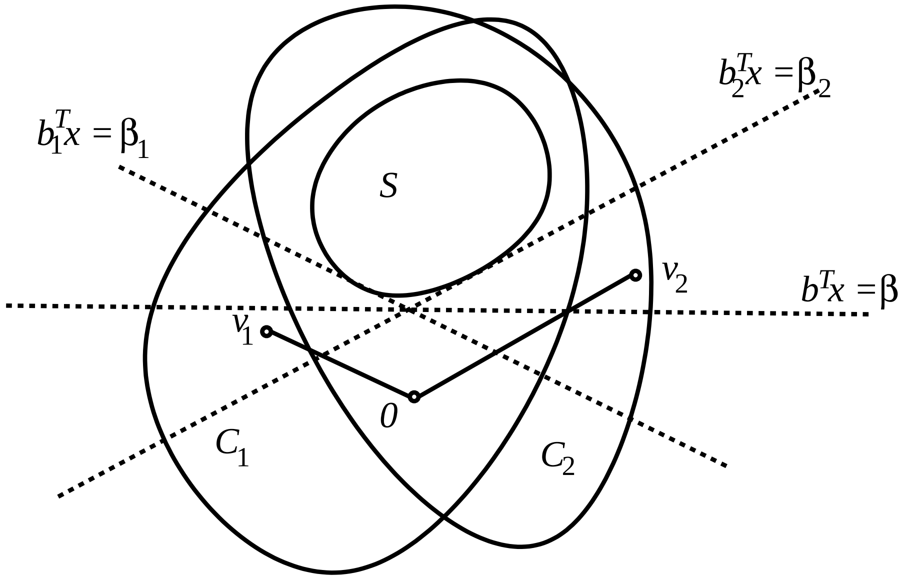
Main Result 1: Separators and Optimality of Embeddings

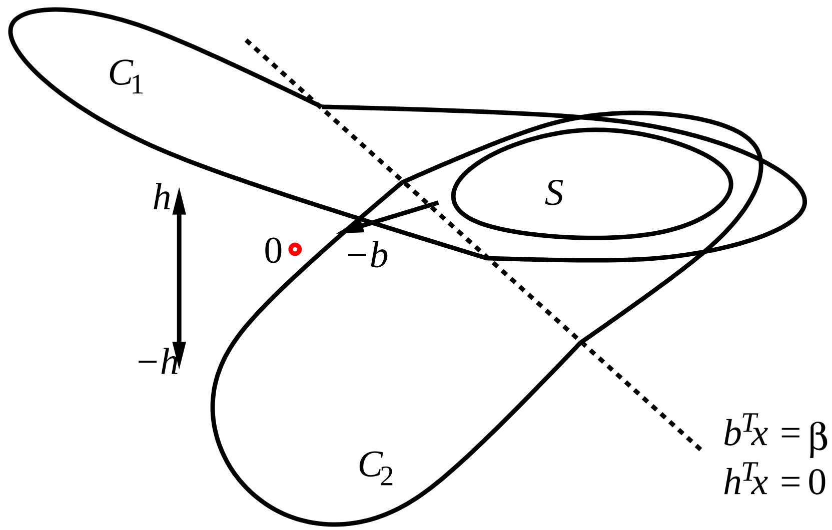
Given optimal $v_i \in \mathbb{R}^n$, $i \in N$, of a connected graph $G = (N, E)$ and a separator $S \subset N$ separating G into node sets C_1, C_2 so that no edges run between C_1 and C_2, let $\mathcal{S} = \{v_i : i \in S\}$.

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Separator-Shadow Th. For at least one $j \in \{1, 2\}$ $[v_i, 0] \cap \operatorname{conv} \mathcal{S} \neq \emptyset \quad \forall i \in C_j$	
geometric proof idea: 	

Sketch of Proof: by contradiction

Assume the Theorem does not hold, then w.l.o.g.
there are points v_1, v_2 with $1 \in C_1$, $2 \in C_2$ and $[0, v_1] \cap \mathcal{S} = [0, v_2] \cap \mathcal{S} = \emptyset$.





$b_i^T x < \beta_i$ separates $[0, v_i]$ from $\mathcal{S}$

choose $\alpha \in [0, 1]$ so that both $C_i \cap \{x : b^T x < \beta\} \neq \emptyset$

$$\begin{bmatrix} b \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} b_1 \\ \beta_1 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} b_2 \\ \beta_2 \end{bmatrix}$$

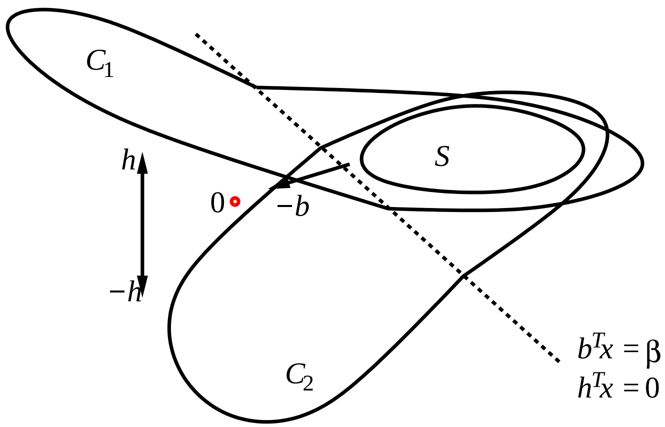
$\sum v_i = 0 \Rightarrow$ lin. dep., thus h exists

Verify: epsilon movement improves solution $\Rightarrow$ contradiction to optimality $\square$

Main Result 1: Separators and Optimality of Embeddings

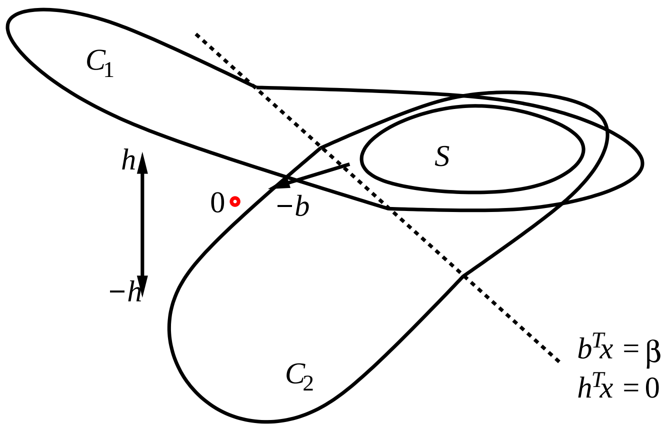
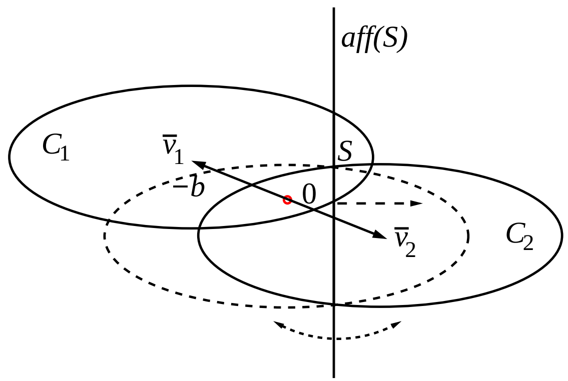
Given optimal $v_i \in \mathbb{R}^n$, $i \in N$, of a connected graph $G = (N, E)$ and a separator $S \subset N$ separating G into node sets C_1, C_2 so that no edges run between C_1 and C_2, let $\mathcal{S} = \{v_i : i \in S\}$.

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Separator-Shadow Th. For at least one $j \in \{1, 2\}$ $$[v_i, 0] \cap \operatorname{conv} \mathcal{S} \neq \emptyset \quad \forall i \in C_j$$	**Separator's Sunny Side Th.** Let $\bar{v}_j = \frac{1}{\lvert C_j \rvert} \sum_{i \in C_j} v_j$, $j \in \{1, 2\}$, be the barycenter of C_j, then $$\bar{v}_j \in \operatorname{aff}(\mathcal{S}) - \operatorname{cone}(\mathcal{S}) \text{ for } j \in \{1, 2\}$$
geometric proof idea:	



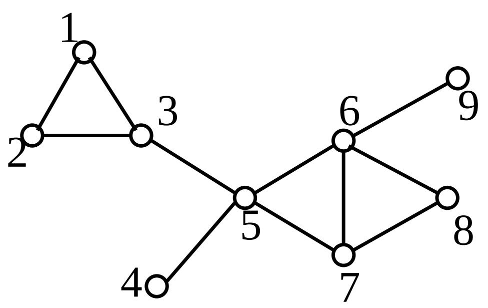
Main Result 1: Separators and Optimality of Embeddings

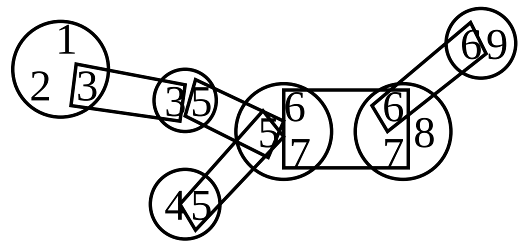
Given optimal $v_i \in \mathbb{R}^n$, $i \in N$, of a connected graph $G = (N, E)$ and a separator $S \subset N$ separating G into node sets C_1, C_2 so that no edges run between C_1 and C_2, let $\mathcal{S} = \{v_i : i \in S\}$.

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Separator-Shadow Th. For at least one $j \in \{1,2\}$ $$[v_i, 0] \cap \operatorname{conv} \mathcal{S} \neq \emptyset \quad \forall i \in C_j$$	**Separator's Sunny Side Th.** Let $\bar{v}_j = \frac{1}{\lvert C_j \rvert} \sum_{i \in C_j} v_j$, $j \in \{1,2\}$, be the barycenter of C_j, then $$\bar{v}_j \in \operatorname{aff}(\mathcal{S}) - \operatorname{cone}(\mathcal{S}) \text{ for } j \in \{1,2\}$$
geometric proof idea: 	geometric proof idea: 

Main Result 2:
Structural bounds on the existence of low dimensional solutions

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Tree-Width Bound There exists an optimal embedding of dimension at most $tw(G)+1$	
needs separator shadow + involved result for separators with $0 \in \operatorname{conv} \mathcal{S}$ algorithmic proof idea: given a tree decomposition, start at a 0-node, try to flatten all adjacent nodes or move on to the next 0-node	



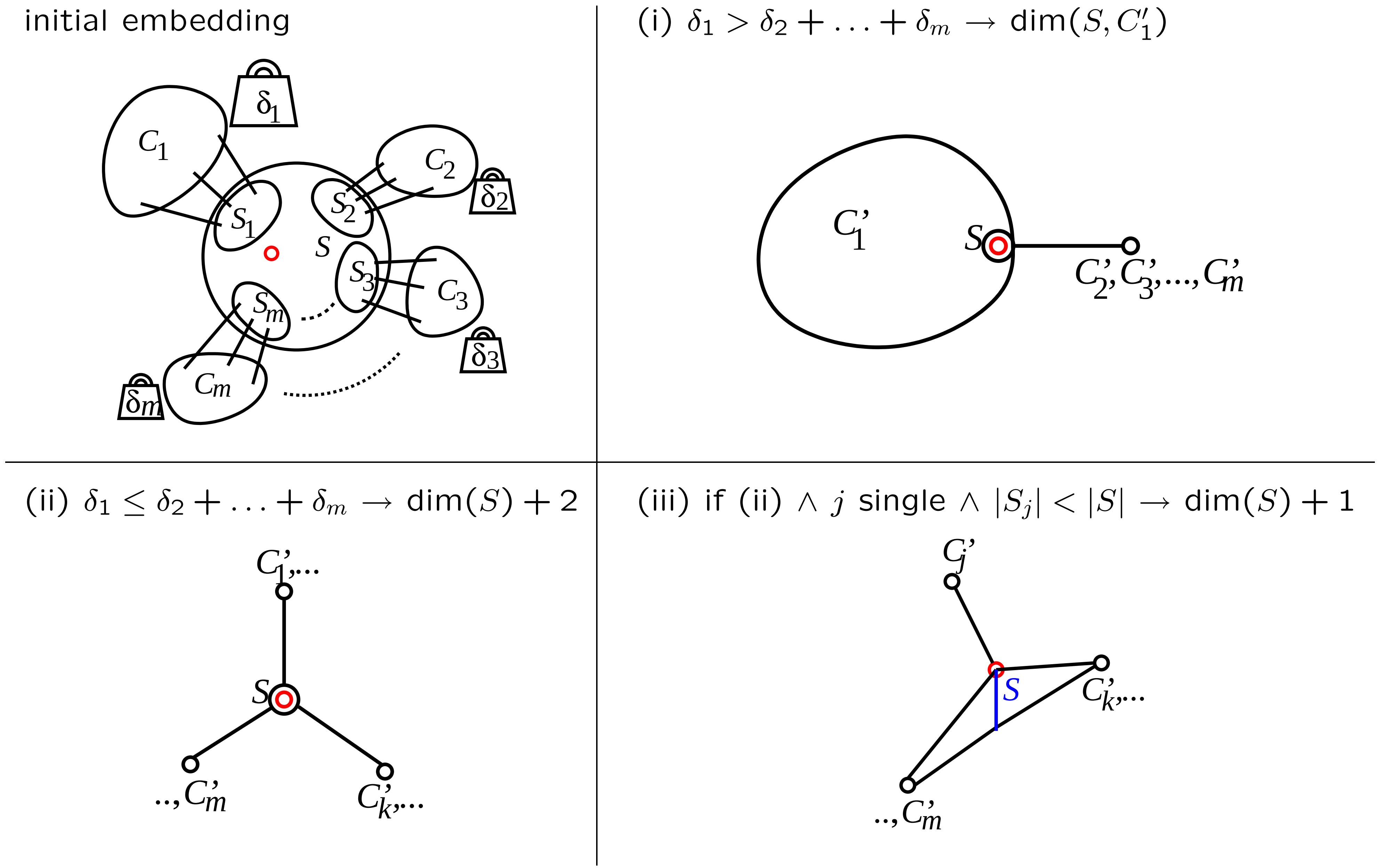


Theorem [Separators Containing the Origin]
Let $v_i \in \mathbb{R}^n$ for $i \in N$ be an optimal solution of (EMB) for a connected graph $G = (N, E)$ and let $S \subset N$ with $0 \in \mathcal{S} = \text{conv}\{v_s : s \in S\}$ be a separator in G inducing a partition $(S, C_1, \ldots, C_m)$ of N so that no node in C_j is adjacent to a node in C_h for $j \neq h$, $j, h \in M = \{1, \ldots, m\}$. Set $\mathcal{L} = \text{span}\,\mathcal{S}$ and, for $j \in M$, $\delta_j = \sum_{i \in C_j} \|p_{\mathcal{L}^\perp}(v_i)\|$.

(i) If $\delta_{\hat{\jmath}} > \sum_{j \in M \setminus \{\hat{\jmath}\}} \delta_j$ for one $\hat{\jmath} \in M$ then there exist $h \in \mathcal{L}^\perp$ and an optimal embedding $v'_i \in \mathbb{R}^n$ of (EMB) with $v'_i = v_i$ for $i \in S$, $v'_i \in \mathcal{L} + \text{span}\,\{h, v_i : i \in C_{\hat{\jmath}}\}$ for $i \in C_{\hat{\jmath}}$ and $v'_i \in \mathcal{L} + \{\delta \sum_{i \in C_{\hat{\jmath}}} v'_i : \delta \geq 0\}$ for $i \in \bigcup_{j \in M \setminus \{\hat{\jmath}\}} C_j$. If, in addition, there exists $\bar{b} \in \text{span}\,\{v_i : i \in C_{\hat{\jmath}}\}$, $\|\bar{b}\| = 1$ so that $\langle \bar{b}, v_i \rangle \geq 0$ for all $i \in C_{\hat{\jmath}}$, then such an embedding exists with $h = 0$.

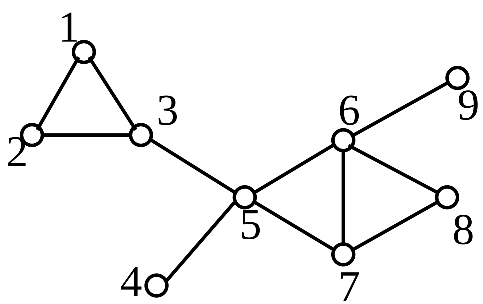
(ii) If $\delta_{\hat{\jmath}} \leq \sum_{j \in M \setminus \{\hat{\jmath}\}} \delta_j$ for all $\hat{\jmath} \in M$ then there exist vectors $d_1, d_2, d_3 \in \mathcal{L}^\perp$, $\|d_1\| = \|d_2\| = \|d_3\| = 1$ with $\dim \text{span}\,\{d_1, d_2, d_3\} \leq 2$, $b_j \in \{d_1, d_2, d_3\}$, $j \in M$, and an optimal embedding $v'_i \in \mathbb{R}^n$, $i \in N$, of (EMB) with $v'_i = v_i$ for $i \in S$ so that for each $j \in M$ we have $v'_i \in \mathcal{L} + \{\delta b_j : \delta \geq 0\}$ for all $i \in C_j$. One may assume $b_j = d_1$ for at most one $j \in M$.

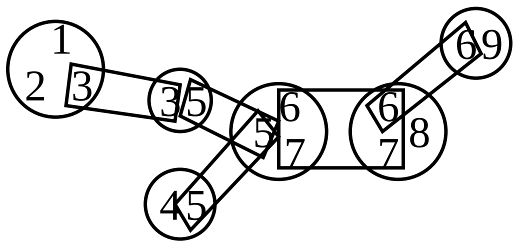
(iii) If, in case (ii), the index $\hat{\jmath} \in M$ is the only $j \in M$ satisfying $b_j = d_1$ and at most $|S| - 1$ nodes of S are adjacent to nodes in $C_{\hat{\jmath}}$, then there is an optimal embedding of dimension at most $|S|$.


initial embedding
δ1
C1
S1
δ2
C2
S2
S
S3
C3
δ3
Sm
Cm
δm
(i) $\delta_1 > \delta_2 + \ldots + \delta_m \rightarrow \dim(S, C'_1)$
C'1
S
C'2, C'3, ..., C'm
(ii) $\delta_1 \leq \delta_2 + \ldots + \delta_m \rightarrow \dim(S) + 2$
C'1, ...
S
.., C'm
C'k, ...
(iii) if (ii) $\wedge$ j single $\wedge$ $|S_j| < |S| \rightarrow \dim(S) + 1$
C'j
S
C'k, ...
.., C'm

Main Result 2:
Structural bounds on the existence of low dimensional solutions

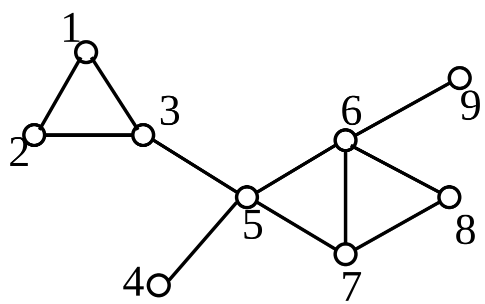
$\lambda_2(L_w)$	$\lambda_{\text{max}}(L_w)$
Tree-Width Bound There exists an optimal embedding of dimension at most $$tw(G) + 1$$	
needs separator shadow + involved result for separators with $0 \in \operatorname{conv} \mathcal{S}$ algorithmic proof idea: given a tree decomposition, start at a 0-node, try to flatten all adjacent nodes or move on to the next 0-node	

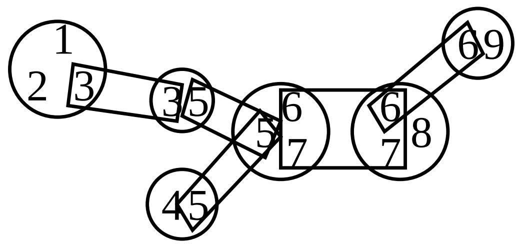




Main Result 2:
Structural bounds on the existence of low dimensional solutions

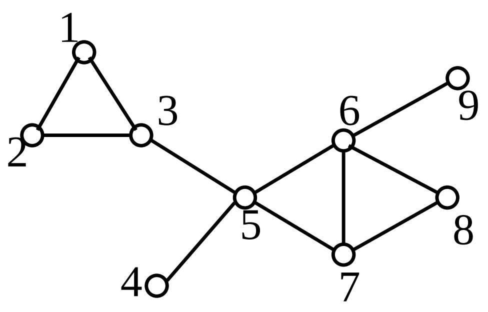
$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Tree-Width Bound There exists an optimal embedding of dimension at most $tw(G)+1$	**Tree-Width Bound** There exists an optimal embedding of dimension at most $tw(G)+1$
needs separator shadow + involved result for separators with $0 \in \operatorname{conv} \mathcal{S}$ algorithmic proof idea: given a tree decomposition, start at a 0-node, try to flatten all adjacent nodes or move on to the next 0-node	

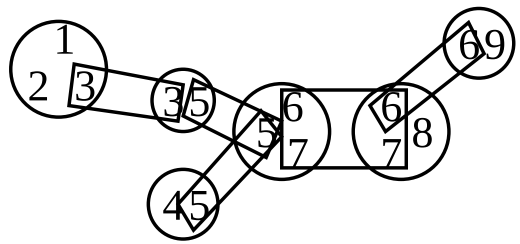




Main Result 2:
Structural bounds on the existence of low dimensional solutions

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
Tree-Width Bound There exists an optimal embedding of dimension at most $$tw(G) + 1$$	**Tree-Width Bound** There exists an optimal embedding of dimension at most $$tw(G) + 1$$
needs separator shadow + involved result for separators with $0 \in \operatorname{conv} \mathcal{S}$ algorithmic proof idea: given a tree decomposition, start at a 0-node, try to flatten all adjacent nodes or move on to the next 0-node	Obs.: in separated sets no forces interact outside separator space. algorithmic proof idea: given a tree decomposition, find a node S with maximal dim(lin($\mathcal{S}$)). For adjacent nodes U, rotate basis of U outside lin($S \cap U$) into lin($\mathcal{S}$), continue recursively



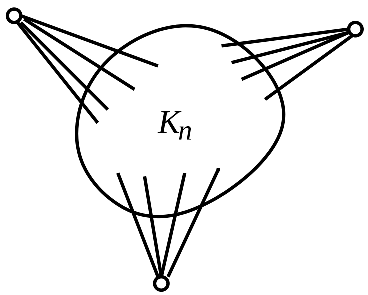


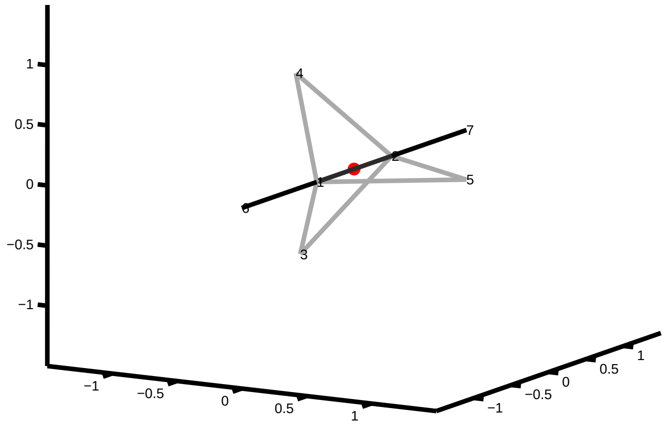
Result 3: Sharpness of Tree-Width Dimension Bounds

$\lambda_2(L_w)$

For $n \geq 4$, connect three vertices completely to K_n

$$tw(G) = n, \quad \text{dim} = n + 1$$

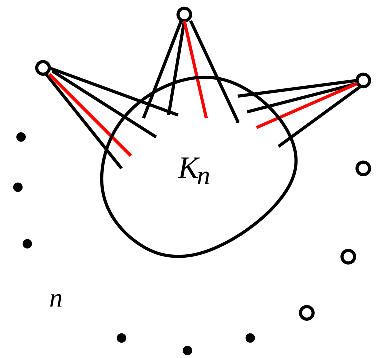


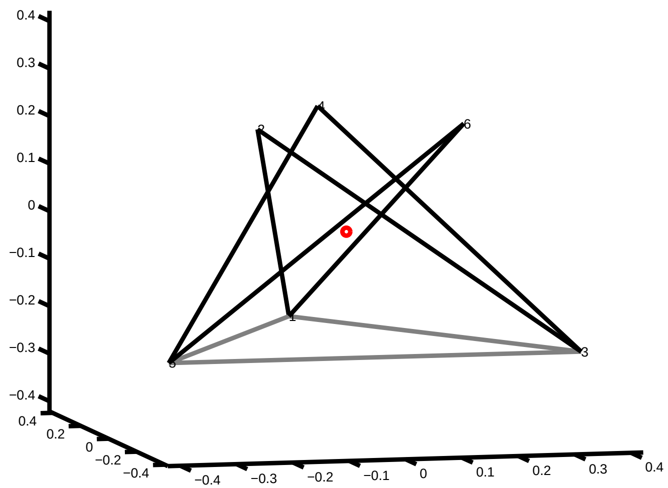


$\lambda_{\max}(L_w)$

Connect n vertices completely to K_n, delete a perfect matching

$$tw(G) = n, \quad \text{dim} = n + 1$$





Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\rightarrow l_{ij}$ for $ij \in E$

Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\to l_{ij}$ for $ij \in E$

To obtain an SDP: bound the edge lengths by $\sum l_{ij}^2 \leq |E|$

$$
(F) \quad \begin{array}{rll}
\max & \sum_{i \in N} \|v_i\|^2 & \\
\text{s.t.} & \|v_i - v_j\| \leq l_{ij} & \text{for } ij \in E, \\
 & \sum_{i \in N} v_i = 0, & \\
 & \sum_{ij \in E} l_{ij}^2 \leq |E|, & \\
 & l \in \mathbb{R}^E,\ v_i \in \mathbb{R}^n \text{ for } i \in N &
\end{array}
$$

Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\rightarrow l_{ij}$ for $ij \in E$

To obtain an SDP: bound the edge lengths by $\sum l_{ij}^2 \leq |E|$

$$
(F) \quad \begin{array}{rll}
\max & \sum_{i\in N} \|v_i\|^2 & \\
\text{s.t.} & \|v_i - v_j\| \leq l_{ij} & \text{for } ij \in E, \\
 & \sum_{i\in N} v_i = 0, & \\
 & \sum_{ij\in E} l_{ij}^2 \leq |E|, & \\
 & l \in \mathbb{R}^E,\ v_i \in \mathbb{R}^n \text{ for } i \in N &
\end{array}
$$

Theorem. For $G = (N, E)$ connected and $V = [v_1, \ldots, v_n]$ optimal for (F), $\sum_{i\in N} \|v_i\|^2 = \frac{|E|}{\lambda_2(L(G))}$ and $V^\top u$ is an eigenvector of $\lambda_2(L(G))$ for $u \in \mathbb{R}^n$.

Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\rightarrow l_{ij}$ for $ij \in E$

To obtain an SDP: bound the edge lengths by $\sum l_{ij}^2 \leq |E|$

$$
(F) \quad \begin{array}{rll}
\max & \sum_{i\in N} \|v_i\|^2 & \\
\text{s.t.} & \|v_i - v_j\| \leq l_{ij} & \text{for } ij \in E, \\
& \sum_{i\in N} v_i = 0, & \\
& \sum_{ij\in E} l_{ij}^2 \leq |E|, & \\
& l \in \mathbb{R}^E,\ v_i \in \mathbb{R}^n \text{ for } i \in N &
\end{array}
$$

$$
\begin{array}{rl}
\min & |E|\rho \\
\text{s.t.} & \sum_{ij\in E} w_{ij} E_{ij} + \mu e e^\top \succeq I, \\
& \rho - w_{ij} = 0 \quad \text{for } ij \in E, \\
& w \in \mathbb{R}^E_+, \rho \geq 0, \mu \in \mathbb{R}
\end{array}
$$

Theorem. For $G = (N, E)$ connected and $V = [v_1, \ldots, v_n]$ optimal for (F), $\sum_{i\in N} \|v_i\|^2 = \frac{|E|}{\lambda_2(L(G))}$ and $V^\top u$ is an eigenvector of $\lambda_2(L(G))$ for $u \in \mathbb{R}^n$.

Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\to l_{ij}$ for $ij \in E$

To obtain an SDP: bound the edge lengths by $\sum l_{ij}^2 \le |E|$

$$(F)\quad \begin{array}{rll} \max & \sum_{i\in N} \|v_i\|^2 & \\ \text{s.t.} & \|v_i - v_j\| \le l_{ij} & \text{for } ij \in E, \\ & \sum_{i\in N} v_i = 0, & \\ & \sum_{ij\in E} l_{ij}^2 \le |E|, & \\ & l \in \mathbb{R}^E,\ v_i \in \mathbb{R}^n \text{ for } i \in N & \end{array}$$

$$\begin{array}{rl} \min & |E|\rho \\ \text{s.t.} & \sum_{ij\in E} w_{ij} E_{ij} + \mu ee^\top \succeq I, \\ & \rho - w_{ij} = 0 \quad \text{for } ij \in E, \\ & w \in \mathbb{R}_+^E, \rho \ge 0, \mu \in \mathbb{R} \end{array}$$

Theorem. For $G = (N, E)$ connected and $V = [v_1, \ldots, v_n]$ optimal for (F), $\sum_{i\in N} \|v_i\|^2 = \frac{|E|}{\lambda_2(L(G))}$ and $V^\top u$ is an eigenvector of $\lambda_2(L(G))$ for $u \in \mathbb{R}^n$.

Theorem. $G = (N, E)$ connected, $u \in \mathbb{R}^n$, $\|u\| = 1$ eigenvector to $\lambda_2(L(G))$, then $X = \frac{|E|}{\lambda_2(L(G))} uu^\top$ and $l_{ij}^2 = \frac{|E|}{\lambda_2(L(G))}(u_i - u_j)^2$, $ij \in E$ is optimal for (F).

Result 4: A Graph Realization for Fiedler Vectors

Fiedler vectors: eigenvectors to $\lambda_2(L(G))$
starting point of spectral graph partitioning heuristics

Idea: Try to spread out the vertices even further by redistributing the edge lengths $\to l_{ij}$ for $ij \in E$

To obtain an SDP: bound the edge lengths by $\sum l_{ij}^2 \leq |E|$

$$(F) \quad \begin{array}{rll} \max & \sum_{i\in N} \|v_i\|^2 & \\ \text{s.t.} & \|v_i - v_j\| \leq l_{ij} & \text{for } ij \in E, \\ & \sum_{i \in N} v_i = 0, & \\ & \sum_{ij\in E} l_{ij}^2 \leq |E|, & \\ & l \in \mathbb{R}^E,\ v_i \in \mathbb{R}^n \text{ for } i \in N & \end{array}$$

$$\begin{array}{rl} \min & |E|\rho \\ \text{s.t.} & \sum_{ij\in E} w_{ij}E_{ij} + \mu ee^\top \succeq I, \\ & \rho - w_{ij} = 0 \quad \text{for } ij \in E, \\ & w \in \mathbb{R}^E_+, \rho \geq 0, \mu \in \mathbb{R} \end{array}$$

Theorem. For $G = (N, E)$ connected and $V = [v_1, \ldots, v_n]$ optimal for (F), $\sum_{i\in N} \|v_i\|^2 = \frac{|E|}{\lambda_2(L(G))}$ and $V^\top u$ is an eigenvector of $\lambda_2(L(G))$ for $u \in \mathbb{R}^n$.

Theorem. $G = (N, E)$ connected, $u \in \mathbb{R}^n$, $\|u\| = 1$ eigenvector to $\lambda_2(L(G))$, then $X = \frac{|E|}{\lambda_2(L(G))} uu^\top$ and $l_{ij}^2 = \frac{|E|}{\lambda_2(L(G))}(u_i - u_j)^2$, $ij \in E$ is optimal for (F).

$\Rightarrow$ Maximum rank optimal solution gives a map of the eigenspace of $\lambda_2(L(G))$.

The same works out for $\lambda_{\max}$, as well.

Generalized λ_2-Problem → Rotational Dimension of a Graph

Given connected $G = (N, E)$, node weights $s_i \geq 0$, edge lengths $l_{ij} \geq 0$,

$$\text{EMB}(s,l) \qquad \begin{array}{rll} \max & \sum_{i \in N} s_i \|v_i\|^2 & \\ \text{s.t.} & \sum_{i \in N} s_i v_i = 0 & \\ & \|v_i - v_j\|^2 \leq l_{ij} & ij \in E \\ & v_i \in \mathbb{R}^n \text{ for } i \in N. & \end{array}$$

Generalized λ_2-Problem $\rightarrow$ Rotational Dimension of a Graph

Given connected $G = (N, E)$, node weights $s_i \geq 0$, edge lengths $l_{ij} \geq 0$,

$$\text{EMB}(s,l) \qquad \begin{array}{rll} \max & \sum_{i\in N} s_i \|v_i\|^2 & \\ \text{s.t.} & \sum_{i\in N} s_i v_i = 0 & \\ & \|v_i - v_j\|^2 \leq l_{ij} & ij \in E \\ & v_i \in \mathbb{R}^n \text{ for } i \in N. & \end{array}$$

Minimal dimension of an optimal solution for weights s and length l

$$\dim_G(s,l) = \min\{\dim \operatorname{span}\{v_i : i \in N\} : v_i \text{ optimal for EMB}(s,l)\}$$

Generalized λ_2-Problem $\to$ Rotational Dimension of a Graph

Given connected $G = (N, E)$, node weights $s_i \geq 0$, edge lengths $l_{ij} \geq 0$,

$$\text{EMB}(s,l) \qquad \begin{array}{rll} \max & \sum_{i \in N} s_i \|v_i\|^2 & \\ \text{s.t.} & \sum_{i \in N} s_i v_i = 0 & \\ & \|v_i - v_j\|^2 \leq l_{ij} & ij \in E \\ & v_i \in \mathbb{R}^n \text{ for } i \in N. & \end{array}$$

Minimal dimension of an optimal solution for weights s and length l

$$\dim_G(s,l) = \min\{\dim \operatorname{span}\{v_i : i \in N\} : v_i \text{ optimal for EMB}(s,l)\}$$

Rotational Dimension of $G = (N, E)$:

- G connected: $\operatorname{rotdim}(G) := \max\{\dim_G(s,l) : s \in \mathbb{Z}_+^N, l \in \mathbb{Z}_+^E\}$
- $G = (\emptyset, \emptyset)$: $\operatorname{rotdim}(G) := -1$
- G not connected: $\operatorname{rotdim}(G) := \max\{\operatorname{rotdim}(C) : C \text{ is a component of } G\}$

Generalized λ_2-Problem $\to$ Rotational Dimension of a Graph

Given connected $G = (N, E)$, node weights $s_i \geq 0$, edge lengths $l_{ij} \geq 0$,

$$\text{EMB}(s,l) \qquad \begin{array}{rll} \max & \sum_{i \in N} s_i \|v_i\|^2 & \\ \text{s.t.} & \sum_{i \in N} s_i v_i = 0 & \\ & \|v_i - v_j\|^2 \leq l_{ij} & ij \in E \\ & v_i \in \mathbb{R}^n \text{ for } i \in N. & \end{array}$$

Minimal dimension of an optimal solution for weights s and length l

$$\dim_G(s,l) = \min\{\dim \operatorname{span}\{v_i : i \in N\} : v_i \text{ optimal for EMB}(s,l)\}$$

Rotational Dimension of $G = (N, E)$:

- G connected: $\operatorname{rotdim}(G) := \max\{\dim_G(s,l) : s \in \mathbb{Z}_+^N, l \in \mathbb{Z}_+^E\}$
- $G = (\emptyset, \emptyset)$ $\operatorname{rotdim}(G) := -1$
- G not connected: $\operatorname{rotdim}(G) := \max\{\operatorname{rotdim}(C) : C \text{ is a component of } G\}$

One can prove (for connected G):

$$\begin{aligned} \operatorname{rotdim}(G) &= \max\{\dim_G(s,l) : s \in \mathbb{R}_+^N, l \in \mathbb{R}_+^E\} \\ &= \max\{\dim_G(s,l) : s \in \mathbb{R}_{++}^N, l \in \mathbb{R}_{++}^E\} \end{aligned}$$

Observation The rotational dimension is a minor monotone graph parameter.

Results for the Rotational Dimension

Theorem [Separator-Shadow]

Let $v_i \in \mathbb{R}^n$, $i \in N$, be optimal for EMB(s, l) for a connected $G = (N, E)$, let $C_1 \dot{\cup} S \dot{\cup} C_2$ partition N so that no node in C_1 is adjacent to a node in C_2. Then, for at least one $j \in \{1, 2\}$, for every $i \in C_j$ the straight line segment $[0, v_i]$ intersects the convex hull of the points in S.

Theorem [Tree-Width]

Given a connected graph $G = (N, E)$ with node weights $s \in \mathbb{R}^N_+$ and edge lengths $l \in \mathbb{R}^E_+$, there exists on optimal solution of EMB(s, l) having dimension at most tree-width of G plus one.

Forbidden Minor Characterizations

- rotdim$(G) \leq 0 \Leftrightarrow$ all components are nodes (forbidden K_2)
- rotdim$(G) \leq 1 \Leftrightarrow$ all components are paths (forbidden K_3, $K_{1,3}$)
- rotdim$(G) \leq 2 \Leftrightarrow$ all components are outerplanar (forbidden K_4, $K_{2,3}$)

Open: rotdim$(G) \leq 3$? [forbidden K_5, but rotdim$(K_{3,3}) = 3$]

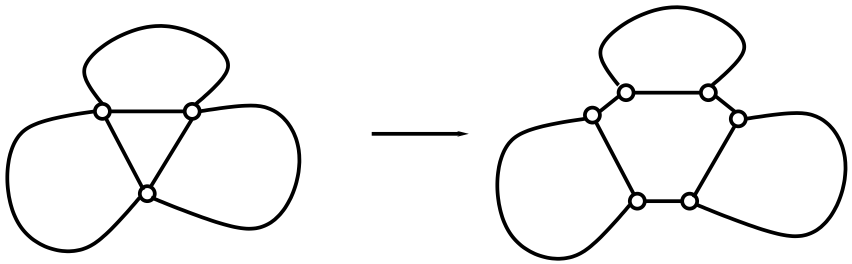
rotdim$(G) \leq 2 \Leftrightarrow$ **all components of G are outerplanar**

Outerplanar graphs are characterized by forbidden minors K_4 and $K_{2,3}$

- rotdim$(K_4) = 3$: regular 3-dim simplex,
- rotdim$(K_{2,3}) = 3$: Let $N = \{1,2\} \cup \{3,4,5\}$ and choose $l_{1i} = 1$ and $l_{2i} = 2$ for $i \in \{3,4,5\}$.

Main idea of rotdim(outerplanar) ≤ 2:

Take a connected outerplanar graph G' and enlarge it to a connected outerplanar graph G having maximum degree ≤ 3:



The former is a minor of the latter, so rotdim$(G') \leq$ rotdim(G).

Given an optimal embedding v_i of G for $s > 0$ and $l > 0$, we show dim span$\{v_i : i \in N\} \leq 2$ by case distinction w.r.t. the position of the origin.

Concluding Remarks

- Results show a clear connection between separator structure and structural properties of the extremal eigenvectors
- For many classes of graphs the tree-width bounds are far from optimal:
  Theorem for $\lambda_{\max}$:
  Any bipartite graph has a 1-dim optimal embedding
- Conjecture for λ_2: planar graphs have 3-dim optimal embeddings
- Relation of Rotational Dimension to the Colin de Verdière number?
  (certainly not equal, but maybe rotdim$\leq \mu$)