ESI Summer School Klagenfurt

Problem sessions

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1. Consider the semidefinite program

$$0 = \min \left\{ x_1 \;\middle|\; \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1+1 \end{pmatrix} \succeq 0 \right\}.$$

   Show that this program violates Slaters condition, and that the duality gap is strictly positive.

2. The system $\dot{x} = Ax$ with

$$A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

   is stable. Let $||B|| := \sum_{i,j} |B_{i,j}|$. We like to find the largest constant α such that $\dot{x}(t) = A(t)x(t)$ is stable whenever $||A(t) - A|| < \alpha$ holds true for all $t \geq 0$.

   Using semidefinite programming and bisection set up a strategy to find a lower bound for α.

3. Consider the barrier function $\phi((x;t)) := -\log(t^2 - x^T x)$ for the "ice-cream-cone" $\{(x;t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\}$. Note that the function "$x^T x - t^2$" is not convex in the variable $(x;t) \in \mathbb{R}^{n+1}$. Show that, nevertheless, ϕ is self-concordant with parameter $\theta = 2$, i.e. (with the definition of l in the lecture) we have

$$l'''(0) \leq 2(l''(0))^{3/2} \quad \text{and} \quad l'(0) \leq 2(l''(0))^{1/2}.$$

4. For given $X, S \succ 0$ we compute the "unsymmetrized" Newton direction (where ΔX is not symmetric) from the system

$$\begin{aligned} \mathcal{A}(X) &= b, \\ \mathcal{A}^*(y) + S &= C, \\ XS &= \mu I, \end{aligned}$$

i.e. we compute

$$\begin{aligned} \mathcal{A}(\Delta X) &= b - \mathcal{A}(X), \\ \mathcal{A}^*(\Delta y) + \Delta S &= C - \mathcal{A}^*(y) - S, \\ X\Delta S + \Delta X S &= \mu I - XS. \end{aligned}$$

We then compute the (unsymmetric) Newton direction for the system where the last line is replaced with $SX = \mu I$.

Finally, we take the arithmetic mean of both. Show that the result coincides with the HKM-direction. (The HKM-direction is obtained by simply symmetrizing the result of the first system).

5. What happens with the APD-algorithm when the underlying SDP or its dual do not have a feasible solution?

6. Write down the necessary and sufficient (local) optimality conditions for the rank-1-formulation of the max-cut-problem.

7. Let X be a symmetric positive semidefinite matrix and

$$X = \begin{pmatrix} U^1 & U^2 \end{pmatrix} \begin{pmatrix} D^1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (U^1)^T \\ (U^2)^T \end{pmatrix}$$

be its eigenvalue decomposition. Show that the tangential cone of the semidefinite cone at X is given by all matrices of the form

$$W = \begin{pmatrix} U^1 & U^2 \end{pmatrix} \begin{pmatrix} * & * \\ * & \Lambda^2 \end{pmatrix} \begin{pmatrix} (U^1)^T \\ (U^2)^T \end{pmatrix}$$

where $\Lambda^2 \succeq 0$ and $*$ can be anything.